

The operator product expansion converges in perturbative field theory

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Abstract

We show, within the framework of the massive Euclidean φ^4 -quantum field theory in four dimensions, that the Wilson operator product expansion (OPE) is not only an asymptotic expansion at short distances as previously believed, but even *converges at arbitrary finite* distances. Our proof rests on a detailed estimation of the remainder term in the OPE, of an arbitrary product of composite fields, inserted as usual into a correlation function with further “spectator fields”. The estimates are obtained using a suitably adapted version of the method of renormalization group flow equations. Convergence follows because the remainder is seen to become arbitrarily small as the OPE is carried out to sufficiently high order, i.e. to operators of sufficiently high dimension. Our results hold for arbitrary, but finite, loop orders. As an interesting side-result of our estimates, we can also prove that the “gradient expansion” of the effective action is convergent.

1 Introduction

All quantum field theories with well-behaved ultra violet behavior are believed to have an operator product expansion (OPE) [18, 19]. This means that the product of any two

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local fields located at nearby points x and y can be expanded in the form

$$\mathcal{O}_A(x)\mathcal{O}_B(y) \sim \sum_C \mathcal{C}_{AB}^C(x-y) \mathcal{O}_C(y), \quad (1)$$

where A, B, C are labels for the various local fields in the given theory (incorporating also their tensor character/spin), and where \mathcal{C}_{AB}^C are certain numerical coefficient functions—or rather distributions—that depend on the theory under consideration, the coupling constants, etc. The sign “ \sim ” indicates that this can be understood as an asymptotic expansion: If the sum on the right side is carried out to a sufficiently large but finite order, then the remainder goes to zero fast as $x \rightarrow y$ in the sense of operator insertions into a quantum state, or into a correlation function. The purpose of this paper is to demonstrate in a specific model that the expansion is not only *asymptotic* in this sense, but even *converges at finite (!) distances*, to arbitrary loop orders, in a perturbative Euclidean quantum field theory.

Our result is not merely a technical footnote, but it furnishes an important insight into the general structure of quantum field theory. Although our result is formulated in a Euclidean setting, this is maybe best explained in the Minkowskian context. There, the analogue of our result would be that correlation functions such as the two-point function $\langle \mathcal{O}_A(x)\mathcal{O}_B(y) \rangle_\Psi$ in a state¹ Ψ are entirely determined by the collection of OPE coefficients which are *state independent*, together with the 1-point functions $\langle \mathcal{O}_C(y) \rangle_\Psi$:

$$\langle \mathcal{O}_A(x)\mathcal{O}_B(y) \rangle_\Psi = \sum_C \mathcal{C}_{AB}^C(x-y) \langle \mathcal{O}_C(y) \rangle_\Psi, \quad (2)$$

where the infinite sum over “ C ” would be convergent, and $(x-y)^2$ would not necessarily have to be small². An analogous statement would apply to the higher n -point functions. Thus, the OPE coefficients capture the state-independent algebraic structure of QFT, while *all* the information about the quantum state, i.e. n -point functions, is contained in the 1-point functions (“form factors”) only. Our result is relevant also in that it supports recent approaches to QFT such as [2, 3, 4] wherein the OPE is taken as the fundamental input.

In this paper, we prove convergence of the OPE in the context of perturbative Euclidean QFT, to arbitrary loop orders. The model that we consider is a hermitian scalar field with self-interaction $g\varphi^4$ and mass $m > 0$ on flat 4-dimensional Euclidean space.

¹The state should have a well-behaved high energy behavior. In the Minkowskian context, it should e.g. have bounded energy E , see below for an appropriate replacement in the Euclidean context.

²Note however that one expects convergence to hold in the relativistic context only for spacelike distances, $(x-y)^2 > 0$, because of locality.

The composite fields \mathcal{O}_A in this model are simply linear combinations of monomials in the basic field φ and its derivatives and are denoted by

$$\mathcal{O}_A = \partial^{w_1} \varphi \cdots \partial^{w_n} \varphi, \quad A = \{n, w\}, \quad (3)$$

where each w_i is a 4-dimensional multi-index, see “notations and conventions” for more on multi-index notation. We define the engineering dimension of such a field as usual by

$$[A] = n + \sum_i |w_i|. \quad (4)$$

Each OPE coefficient $\mathcal{C}_{AB}^C(x-y)$ is itself a formal power series in \hbar (“loop expansion”). As usual in perturbation theory, we will not be concerned with the convergence of these expansions in \hbar . Instead, in this paper, we will be concerned with the convergence of the OPE (i.e. the expansion in “ C ”) at arbitrary but fixed order l in \hbar .

To analyze this issue, we must insert the left- and right sides of (1) into a correlation function containing suitable “spectator fields”, which play the role of a quantum state in the Euclidean context. A simple and natural choice for the spectator fields is e.g.

$$\varphi(f_{p_i}) := \int d^4x \, \varphi(x) f_{p_i}(x), \quad (5)$$

where p_i is a 4-momentum, and where f_{p_i} is a smooth function whose Fourier transform $\hat{f}_{p_i}(q)$ has compact support for q in a ball of radius ϵ around p_i . Our main result is the following

Theorem: Let the sum \sum_C in the operator product expansion (1) be over all C such that

$$[C] - [A] - [B] \leq \Delta \quad (6)$$

where Δ is some positive integer. Then for each such Δ , we have the following bound for the “remainder” in the OPE *in loop order l* :

$$\begin{aligned} & \left| \left\langle \mathcal{O}_A(x) \mathcal{O}_B(0) \varphi(f_{p_1}) \cdots \varphi(f_{p_n}) \right\rangle - \sum_C \mathcal{C}_{AB}^C(x) \left\langle \mathcal{O}_C(0) \varphi(f_{p_1}) \cdots \varphi(f_{p_n}) \right\rangle \right| \quad (7) \\ & \leq m^{[A]+[B]+n} \sqrt{[A]![B]!} \tilde{K}^{[A]+[B]} \prod_i \sup |\hat{f}_{p_i}| \\ & \times \sup(1, \frac{|\vec{p}|_n}{m})^{2([A]+[B])(n+2l+1)+3n} \sum_{\lambda=0}^{n/2+2l} \frac{\log^\lambda \sup(1, \frac{|\vec{p}|_n}{m})}{2^\lambda \lambda!} \\ & \times \frac{1}{\sqrt{\Delta!}} \left(\tilde{K} m |x| \sup(1, \frac{|\vec{p}|_n}{m})^{n+2l+1} \right)^\Delta. \end{aligned}$$

Here, $\langle . \rangle$ denote correlation functions, and \tilde{K} is a constant depending on n, l . Furthermore, $|\vec{p}|_n$ is defined in eq. (51), and f_{p_i} are smooth test functions in position space, whose support in momentum space is contained in a ball of radius ϵ around p_i .

This result establishes the convergence of the OPE, i.e. the sum over C , at each fixed order in perturbation theory, because the remainder evidently goes to zero as $\Delta \rightarrow \infty$. There are no conditions on x , so the OPE converges even at arbitrarily large distances! But we note that such conditions could arise if we were to allow a wider class of spectator fields, for example, if we were to replace f_{p_i} by test-functions whose Fourier transforms are only decaying in momentum space, but are not of compact support. This type of behavior can be understood in a way by the fact that $|\vec{p}|_n$ gives a measure for the “typical energy” of the “state” in which we try to carry out the OPE. As the high energy behavior of the “state” becomes worse, so do the convergence properties of the OPE.

To prove the theorem, one first has to give a prescription for defining the Schwinger functions and OPE coefficients in renormalized perturbation theory. There are several options; in this paper we find it convenient to use the Wilson-Wegner-Polchinski flow equation method [15, 17, 18]. In this method, one first introduces an infrared cutoff called Λ , and an ultraviolet cutoff called Λ_0 . One then defines the quantities of interest for finite values of the cutoffs, and derives for them a flow equation as a function of Λ . For suitable boundary conditions its solutions may be bounded inductively and uniformly in the ultraviolet cutoff Λ_0 . The last fact makes it possible to remove the cutoff³ and at the same time provides non-trivial bounds. In our case, we need bounds for the remainder in the OPE. Again, such bounds are verified inductively.

While the general strategy is rather clear conceptually, it gets more involved in practice. This is because a relatively refined induction hypothesis is required to ensure that it replicates itself in the induction process. The verification of the induction step is thus the main technical task of this paper.

A side result of our estimations which may be of some interest is that the “gradient expansion” (68) of the effective action converges at each fixed number of loops; the precise statement may be found in Cor. 3.1.

³To show not only boundedness but also convergence in the limit $\Lambda_0 \rightarrow \infty$, one also has to study a version of the flow equation that is differentiated w.r.t. the cutoff. We do not perform this step here since it has already been performed in the literature for all quantities of interest in [7, 8]. The bounds obtained there were less precise than those obtained here but this does not matter because also such less stringent bounds are sufficient to merely show convergence in Λ_0 .

Notations and conventions: Our convention for the Fourier transform in \mathbb{R}^4 is

$$f(x) = \int_p \hat{f}(p) e^{ipx} := \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} e^{ipx} \hat{f}(p) . \quad (8)$$

We also use a standard multi-index notation. Our multi-indices are elements $w = (w_1, \dots, w_n) \in \mathbb{N}^{4n}$, so that each $w_i \in \mathbb{N}^4$ is a four-dimensional multiindex whose entries are $w_{i,\mu} \in \mathbb{N}$ and $\mu = 1, \dots, 4$. If $f(\vec{p})$ is a smooth function on \mathbb{R}^{4n} , we set

$$\partial^w f(\vec{p}) = \prod_{i,\mu} \left(\frac{\partial}{\partial p_{i,\mu}} \right)^{w_{i,\mu}} f(\vec{p}) \quad (9)$$

and

$$w! = \prod_{i,\mu} w_{i,\mu}!, \quad |w| = \sum_{i,\mu} w_{i,\mu} . \quad (10)$$

We often need to take derivatives ∂^w of a product of functions $f_1 \dots f_n$. Using the Leibniz rule, such derivatives get distributed over the factors resulting in the sum of all terms of the form $c_{\{v_i\}} \partial^{v_1} f_1 \dots \partial^{v_r} f_r$, where each v_i is now a $4n$ -dimensional multi-index, where $v_1 + \dots + v_r = w$, and where

$$c_{\{v_i\}} = \frac{(v_1 + \dots + v_r)!}{v_1! \dots v_r!} \leq r^{|w|} \quad (11)$$

is the associated weight factor.

If $F(\varphi)$ is a differentiable function (in the Frechet space sense) of the Schwartz space function $\varphi \in \mathcal{S}(\mathbb{R}^4)$, we denote its functional derivative as

$$\frac{d}{dt} F(\varphi + t\psi)|_{t=0} = \int d^4x \frac{\delta F(\varphi)}{\delta \varphi(x)} \psi(x) , \quad \psi \in \mathcal{S}(\mathbb{R}^4) , \quad (12)$$

where the right side is understood in the sense of distributions in $\mathcal{S}'(\mathbb{R}^4)$. Multiple functional derivatives are denoted in a similar way and define in general distributions on multiple cartesian copies of \mathbb{R}^4 .

2 Basic setup, flow equation framework

In this section we introduce the quantities of interest in this paper, namely the Schwinger functions $\langle \dots \rangle$, and the OPE coefficient functions, \mathcal{C}_{AB}^C . For this purpose, we will also derive various useful auxiliary quantities such as connected and amputated Schwinger functions, as well as certain “normal products”. The reason for defining these is that they satisfy a suitably simple version of the flow equations, which we also give below.

Renormalization theory based on the flow equation (FE) [17, 18, 15] of the renormalization group has been reviewed quite often in the literature, so we will be relatively brief. The first presentation in the form we use it here is in [9]. Reviews are in [14] and in [13] (in German).

2.1 Connected amputated Green functions (CAG's)

To begin, we introduce an infrared⁴ cutoff Λ , and an ultraviolet cutoff Λ_0 . These cutoffs enter the definition of the theory through the propagator C^{Λ, Λ_0} which is defined in momentum space by

$$C^{\Lambda, \Lambda_0}(p) = \frac{1}{p^2 + m^2} \left[\exp\left(-\frac{p^2 + m^2}{\Lambda_0^2}\right) - \exp\left(-\frac{p^2 + m^2}{\Lambda^2}\right) \right]. \quad (13)$$

The full propagator is recovered for $\Lambda \rightarrow 0$ and $\Lambda_0 \rightarrow \infty$, and we always assume

$$0 < \Lambda, \quad \kappa := \sup(\Lambda, m) < \Lambda_0. \quad (14)$$

Other choices of regularization are of course admissible. The one chosen in (13) has the advantage of being analytic in p^2 for $\Lambda > 0$. The propagator defines a corresponding Gaussian measure μ^{Λ, Λ_0} , whose covariance is $\hbar C^{\Lambda, \Lambda_0}$. The factor of \hbar is inserted to obtain a consistent loop expansion in the following. The interaction is taken to be

$$L^{\Lambda_0}(\varphi) = \int d^4x \left(a^{\Lambda_0} \varphi(x)^2 + b^{\Lambda_0} \partial\varphi(x)^2 + c^{\Lambda_0} \varphi(x)^4 \right). \quad (15)$$

It contains suitable counter terms satisfying $a^{\Lambda_0} = O(\hbar)$, $b^{\Lambda_0} = O(\hbar^2)$ and $c^{\Lambda_0} = \frac{g}{4!} + O(\hbar)$. They will be adjusted—and actually diverge—when $\Lambda_0 \rightarrow \infty$ in order to obtain a well defined limit of the quantities of interest for us. We have anticipated this by making them “running couplings”, i.e. functions of the ultra violet cutoff Λ_0 . The correlation (= Schwinger-) functions of n basic fields with cutoff are then given by

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle := (Z^{\Lambda, \Lambda_0})^{-1} \int d\mu^{\Lambda, \Lambda_0} \exp\left(-\frac{1}{\hbar} L^{\Lambda_0}\right) \varphi(x_1) \cdots \varphi(x_n). \quad (16)$$

This is just the standard Euclidean path-integral, but note that the free part in the Lagrangian has been absorbed into the Gaussian measure $d\mu^{\Lambda, \Lambda_0}$. The normalization factor is chosen so that $\langle 1 \rangle = 1$. This factor is finite only as long as we impose an additional volume cutoff. But the infinite volume limit can be taken without difficulty once

⁴Such a cutoff is of course not necessary in a massive theory. The IR behavior is substantially modified only for Λ above m .

we pass to perturbative connected correlation functions which we will do in a moment. For more details on this limit see [11, 14]. The path integral will be analyzed in the perturbative sense, i.e. the exponentials are expanded out and the Gaussian integrals are then performed. The full theory is obtained by sending the cutoffs $\Lambda_0 \rightarrow \infty$ and $\Lambda \rightarrow 0$, for a suitable choice of the running couplings. In the flow equation technique, the correct behavior of the running couplings, necessary for a well-defined limit, is obtained by deriving first a differential equation for the Schwinger functions in Λ , and by then defining the running couplings implicitly through the boundary conditions for this equation.

These flow equations are written more conveniently in terms of the hierarchy of “connected, amputated Schwinger functions” (CAG’s). Their generating functional is defined through the convolution⁵ of the Gaussian measure with the exponentiated interaction.

$$-L^{\Lambda, \Lambda_0} := \hbar \log \mu^{\Lambda, \Lambda_0} \star \exp \left(-\frac{1}{\hbar} L^{\Lambda_0} \right) - \hbar \log Z^{\Lambda, \Lambda_0} . \quad (17)$$

The functional L^{Λ, Λ_0} has an expansion as a formal power series in terms of Feynman diagrams with precisely l loops, n external legs, and propagator $C^{\Lambda, \Lambda_0}(p)$. As the name suggests, only connected diagrams contribute, and the (free) propagators on the external legs are removed. We will not use decompositions in terms of Feynman diagrams. But we will also analyze the functional (17) in the sense of formal power series, i.e. we consider the terms in the formal power series

$$L^{\Lambda, \Lambda_0}(\varphi) := \sum_{n>0} \sum_{l=0}^{\infty} \hbar^l \int d^4 x_1 \dots d^4 x_n \mathcal{L}_{n,l}^{\Lambda, \Lambda_0}(x_1, \dots, x_n) \varphi(x_1) \dots \varphi(x_n), \quad (18)$$

where $\varphi \in \mathcal{S}(\mathbb{R}^4)$ is any Schwartz space function. No statement is made about the convergence of the series in \hbar . The objects on the right side, the CAG’s, are the basic quantities in our analysis because they are easier to work with than the full Schwinger functions. But the latter can of course be recovered from the CAG’s.

Because the connected amputated functions in position space are translation invariant, their Fourier transforms, denoted $\mathcal{L}_{n,l}^{\Lambda, \Lambda_0}(p_1, \dots, p_n)$, are supported at $p_1 + \dots + p_n = 0$. We consequently write, by abuse of notation

$$\mathcal{L}_{n,l}^{\Lambda, \Lambda_0}(p_1, \dots, p_n) = \delta^4 \left(\sum_{i=1}^n p_i \right) \mathcal{L}_{n,l}^{\Lambda, \Lambda_0}(p_1, \dots, p_{n-1}), \quad (19)$$

i.e. one of the momenta is determined in terms of the remaining $n - 1$ independent momenta by momentum conservation. It is straightforward to see that, as functions of these

⁵The convolution is defined in general by $(\mu^{\Lambda, \Lambda_0} \star F)(\varphi) = \int d\mu^{\Lambda, \Lambda_0}(\varphi') F(\varphi + \varphi')$.

remaining independent momenta, the connected amputated Green functions are smooth, $\mathcal{L}_{n,l}^{\Lambda,\Lambda_0}(p_1, \dots, p_{n-1}) \in C^\infty(\mathbb{R}^{4(n-1)})$. It is much less obvious, but will be demonstrated later in Cor. 3.1, that they are in fact even analytic functions near $\vec{p} = 0$, even after the cutoffs are removed.

The flow equations are obtained by taking a Λ -derivative of eq.(17):

$$\partial_\Lambda L^{\Lambda,\Lambda_0} = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \right\rangle L^{\Lambda,\Lambda_0} - \frac{1}{2} \left\langle \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} L^{\Lambda,\Lambda_0} \right\rangle + \hbar \partial_\Lambda \log Z^{\Lambda,\Lambda_0} . \quad (20)$$

Here we use the shorthand \dot{C}^Λ for $\partial_\Lambda C^{\Lambda,\Lambda_0}$, which, as we note, does not depend on Λ_0 . By $\langle \cdot, \cdot \rangle$ we denote the standard scalar product in $L^2(\mathbb{R}^4, d^4x)$, and \star denotes convolution in \mathbb{R}^4 . For example

$$\left\langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta\varphi} \right\rangle = \int d^4x d^4y \dot{C}^\Lambda(x-y) \frac{\delta}{\delta\varphi(x)} \frac{\delta}{\delta\varphi(y)} \quad (21)$$

is the “functional Laplace operator”. When expanded out in φ , the flow equations (20) read in momentum space

$$\begin{aligned} \partial_\Lambda \mathcal{L}_{2n,l}^{\Lambda,\Lambda_0}(p_1, \dots, p_{2n-1}) &= \binom{2n+2}{2} \int_k \dot{C}^\Lambda(k) \mathcal{L}_{2n+2,l-1}^{\Lambda,\Lambda_0}(k, -k, p_1, \dots, p_{2n-1}) \\ &- 2 \sum_{\substack{l_1+l_2=l, \\ n_1+n_2=n+1}} n_1 n_2 \mathbb{S} \left[\mathcal{L}_{2n_1,l_1}^{\Lambda,\Lambda_0}(p_1, \dots, p_{2n_1-1}) \dot{C}^\Lambda(q) \mathcal{L}_{2n_2,l_2}^{\Lambda,\Lambda_0}(-q, p_{2n_1}, \dots, p_{2n-1}) \right] \end{aligned} \quad (22)$$

$$\text{with } q = -p_1 - \dots - p_{2n_1-1} = p_{2n_1} + p_{2n_1+1} + \dots + p_{2n} .$$

The symbol \mathbb{S} is an operator which acts on the functions of momenta (p_1, \dots, p_{2n}) by taking the mean value over those permutations π of $(1, \dots, 2n)$, for which $\pi(1) < \pi(2) < \dots < \pi(2n_1 - 1)$ and $\pi(2n_1) < \pi(2n_1 + 1) < \dots < \pi(2n)$. And we used the fact that for the theory proposed through (15), only even moments of the effective action will be non-vanishing due to the symmetry $\varphi \rightarrow -\varphi$, and we thus wrote the equations only for those.

We will also need the FE’s differentiated w.r.t. to components of the momentum variables.

We obtain⁶:

$$\begin{aligned} \partial_\Lambda \partial^w \mathcal{L}_{2n,l}^{\Lambda,\Lambda_0}(p_1, \dots, p_{2n-1}) &= \binom{2n+2}{2} \int_k \dot{C}^\Lambda(k) \partial^w \mathcal{L}_{2n+2,l-1}^{\Lambda,\Lambda_0}(k, -k, p_1, \dots, p_{2n-1}) \\ &- 2 \sum_{\substack{l_1+l_2=l \\ w_1+w_2+w_3=w \\ n_1+n_2=n+1}} n_1 n_2 c_{\{w_j\}} \mathbb{S} \left[\partial^{w_1} \mathcal{L}_{2n_1,l_1}^{\Lambda,\Lambda_0}(p_1, \dots, p_{2n_1-1}) \partial^{w_3} \dot{C}^\Lambda(q) \partial^{w_2} \mathcal{L}_{2n_2,l_2}^{\Lambda,\Lambda_0}(-q, p_{2n_1}, \dots, p_{2n-1}) \right]. \end{aligned} \quad (23)$$

To define the CAG's through the flow equations, we have to impose boundary conditions. These are⁷, using the multi-index convention introduced above in “notations and conventions”:

$$\partial^w \mathcal{L}_{n,l}^{0,\Lambda_0}(\vec{0}) = \delta_{w,0} \delta_{n,4} \delta_{l,0} \frac{g}{4!} \quad \text{for } n + |w| \leq 4, \quad (24)$$

as well as

$$\partial^w \mathcal{L}_{n,l}^{\Lambda_0,\Lambda_0}(\vec{p}) = 0 \quad \text{for } n + |w| > 4. \quad (25)$$

The CAG's are then determined by integrating the flow equations subject to these boundary conditions, see [9, 14]. In our context this is described in detail when we come to the estimates of the CAG's in sec. 3.

2.2 Insertions of composite fields, normal products and OPE coefficients

For the purposes of this paper, and also in many applications, one would like to define not only Schwinger functions of products of the basic field, but also ones containing composite operators. These are obtained by replacing the action L^{Λ_0} with an action containing additional sources. To set things up properly, it is useful to introduce first some notation. Let $\mathcal{F}_{\text{loc}}^\infty$ be the space of smooth local, polynomial functionals $F(\varphi)$ of $\varphi \in \mathcal{S}(\mathbb{R}^4)$. Any such functional can be written by definition as

$$F(\varphi) = \sum_A \int d^4x \mathcal{O}_A(x) f^A(x), \quad f^A \in C_0^\infty(\mathbb{R}^4), \quad (26)$$

⁶In distributing the derivatives over the three factors in the second term on the r.h.s. with the Leibniz rule, we have tacitly assumed that the momentum p_i appears among those from $\mathcal{L}_{2n_1,l_1}^{\Lambda,\Lambda_0}$. If this is not the case one has to parametrize $\mathcal{L}_{2n_1,l_1}^{\Lambda,\Lambda_0}$ in terms of (say) $(p_2, \dots, p_{2n_1-1}, q)$ with $q = p_{2n_1} + \dots + p_{2n}$, in order to introduce the p_i -dependence in $\mathcal{L}_{2n_1,l_1}^{\Lambda,\Lambda_0}$. For a fully systematic treatment see [1].

⁷We restrict to BPHZ renormalization conditions in their simplest form, more general choices are of course equally admissible.

where \mathcal{O}_A are composite operators as in eq. (3) and where the sum is finite. We now consider instead of L^{Λ_0} a modified action containing sources f^A , given by replacing

$$L^{\Lambda_0} \rightarrow L_F^{\Lambda_0} := L^{\Lambda_0} + F + \sum_{j=0}^{\infty} B_j^{\Lambda_0}(\underbrace{F \otimes \cdots \otimes F}_j), \quad (27)$$

where the last term represents the counter terms and is for each j a suitable linear functional⁸

$$B_j^{\Lambda_0} : (\mathcal{F}_{\text{loc}}^{\infty})^{\otimes j} \rightarrow \mathcal{F}^{\infty}, \quad (28)$$

that is symmetric, and of order $O(\hbar)$. These counter terms are designed to eliminate the divergences arising from composite field insertions in the Schwinger functions when one takes $\Lambda_0 \rightarrow \infty$. The Schwinger functions with insertions of r composite operators are defined with the aid of functional derivatives with respect to the sources, setting the sources $f^{A_i} = 0$ afterwards:

$$\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_r}(x_r) \rangle := \hbar^r \frac{\delta^r}{\delta f^{A_1}(x_1) \cdots \delta f^{A_r}(x_r)} (Z^{\Lambda, \Lambda_0})^{-1} \int d\mu^{\Lambda, \Lambda_0} \exp \left(-\frac{1}{\hbar} L_F^{\Lambda_0}(\varphi) \right) \Big|_{f^{A_i}=0}. \quad (29)$$

The previous definition of the CAG's is a special case of this; there we take $F = \int d^4x f(x) \varphi(x)$, and we have $B_j^{\Lambda_0}(F^{\otimes j}) = 0$, because no extra counter terms are required for this simple insertion. As above, we can define a corresponding effective action as

$$-L_F^{\Lambda, \Lambda_0} := \hbar \log \mu^{\Lambda, \Lambda_0} \star \exp \left(-\frac{1}{\hbar} (L^{\Lambda_0} + F + \sum_{j=0}^{\infty} B_j^{\Lambda_0}(F^{\otimes j})) \right) - \log Z^{\Lambda, \Lambda_0} \quad (30)$$

which is now a functional of the sources f^{A_i} , as well as of φ . Differentiating r times with respect to the sources, and setting them to zero afterwards, gives the generating functionals of the CAG's with r operator insertions, namely:

$$L^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(x_1) \otimes \cdots \otimes \mathcal{O}_{A_r}(x_r)) = \frac{\delta^r L_F^{\Lambda, \Lambda_0}}{\delta f^{A_1}(x_1) \cdots \delta f^{A_r}(x_r)} \Big|_{f^{A_i}=0}. \quad (31)$$

The CAG's with insertions satisfy a number of obvious properties, e.g. they are multi-linear—as indicated by the tensor product notation—and symmetric in the insertions.

⁸As we will see it is possible to impose boundary conditions such that the multiply inserted Schwinger functions become less singular at short distances $(x_i - x_j)^2 \rightarrow 0$. In this case the maps B_j take their values in the space \mathcal{F}^{∞} of *non-local* functionals on Schwartz space.

As the CAG's without insertions, the CAG's with insertions can be further expanded in φ and \hbar , and this is denoted as

$$L^{\Lambda, \Lambda_0} \left(\bigotimes_{i=1}^r \mathcal{O}_{A_i}(x_i) \right) = \sum_{n, l \geq 0} \hbar^l \int d^4 y_1 \dots d^4 y_n \mathcal{L}_{n, l}^{\Lambda, \Lambda_0} \left(\bigotimes_{i=1}^r \mathcal{O}_{A_i}(x_i); y_1, \dots, y_n \right) \prod_{j=1}^n \varphi(y_j).$$

Due to the insertions in $\mathcal{L}_{n, l}^{\Lambda, \Lambda_0}(\otimes_j \mathcal{O}_{A_j}(x_j), \vec{p})$, there is no restriction on the momentum set \vec{p} . However it follows from translation invariance that functions with insertions at a translated set of points $x_j + a$ are obtained from those at $a = 0$ upon multiplication by

$$e^{ia \sum_{i=1}^n p_i} . \quad (32)$$

The CAG's with insertions satisfy a flow equation of a similar nature as those without insertions, and these are again obtained by taking derivatives of eq. (20) with respect to the sources. For example, for one insertion, the flow equation (FE) is:

$$\partial_\Lambda L^{\Lambda, \Lambda_0}(\mathcal{O}_A) = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta \varphi} \right\rangle L^{\Lambda, \Lambda_0}(\mathcal{O}_A) - \left\langle \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0}(\mathcal{O}_A), \dot{C}^\Lambda \star \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0} \right\rangle + \partial_\Lambda \log Z^{\Lambda, \Lambda_0} . \quad (33)$$

It is important to note that this FE is *linear and homogeneous* w.r.t. the functional $L^{\Lambda, \Lambda_0}(\mathcal{O}_A)$. The FE's for multiple insertions are obtained similarly by taking more functional derivatives with respect to the sources, for example for two insertions:

$$\begin{aligned} \partial_\Lambda L^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) &= \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \varphi}, \dot{C}^\Lambda \star \frac{\delta}{\delta \varphi} \right\rangle L^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) \\ &- \left\langle \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B), \dot{C}^\Lambda \star \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0} \right\rangle \\ &- \left\langle \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0}(\mathcal{O}_A), \dot{C}^\Lambda \star \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0}(\mathcal{O}_B) \right\rangle + \partial_\Lambda \log Z^{\Lambda, \Lambda_0} . \end{aligned} \quad (34)$$

Thus, the flow equation for the CAG's with two insertions is not linear homogeneous, but involves a “source term” which is quadratic in the CAG's with one insertion. If we want to integrate the flow equations with insertions, we therefore have to ascend in the number of insertions.

Expanding the FE's for the generating functionals in terms of \hbar and φ gives us again a corresponding hierarchy of FE's satisfied by the $\mathcal{L}_{n, l}^{\Lambda, \Lambda_0}(\otimes_i \mathcal{O}_{A_i}; \vec{p})$. For one insertion, these equations are given below in eq. (47), whereas for two insertions, they are given below in eq. (48) (without the index “D”).

The CAG's with one insertion are not uniquely defined without imposing suitable boundary conditions on the corresponding FE. For an operator \mathcal{O}_A and $A = \{n', w'\}$ (so

that its dimension is $[A] = n' + |w'|$) the simplest choice of boundary conditions, which also goes under the name of "normal product", is

$$\partial^w \mathcal{L}_{n,l}^{\Lambda_0, \Lambda_0}(\mathcal{O}_A(0); \vec{p}) = 0 \quad \text{for } n + |w| > n' + |w'|, \quad (35)$$

and

$$\partial^w \mathcal{L}_{n,l}^{0, \Lambda_0}(\mathcal{O}_A(0); 0) = i^{|w|} w! \delta_{w,w'} \delta_{n,n'} \delta_{l,0} \quad \text{for } n + |w| \leq n' + |w'|. \quad (36)$$

The δ -symbol only depends on the sets $\{w\} = \{w_1, \dots, w_n\}$ and $\{w'\} = \{w'_1, \dots, w'_{n'}\}$. Due to the linearity of the FE, linear superpositions of normal products are also solutions of the system of FE, and their boundary values are the corresponding superpositions.

It is also possible to extend the definition of the normal products in the following sense which leads to the appearance of an additional index, D , measuring the degree of regularity. For one insertion, these "oversubtracted" normal products are denoted $\mathcal{L}_{n,l,D}^{\Lambda, \Lambda_0}(\mathcal{O}_A; \vec{p})$ and are defined for any $D \geq [A]$ through

$$\partial^w \mathcal{L}_{n,l,D}^{\Lambda_0, \Lambda_0}(\mathcal{O}_A(0); \vec{p}) = 0 \quad \text{for } n + |w| > D, \quad (37)$$

and

$$\partial^w \mathcal{L}_{n,l,D}^{0, \Lambda_0}(\mathcal{O}_A(0); \vec{0}) = m^{D-n-|w|} i^{|w|} w! \delta_{w,w'} \delta_{n,n'} \delta_{l,0} \quad \text{for } n + |w| \leq D. \quad (38)$$

In particular, for $D = n' + |w'| = [A]$, the oversubtracted normal products agree with the previous ones, because they then satisfy the same FE and the same boundary conditions.

As the CAG's with one insertion, the CAG's with multiple insertions are not uniquely defined by the FE without imposing a boundary condition. The simplest boundary conditions for two insertions are given by:

$$\partial^w \mathcal{L}_{n,l}^{\Lambda_0, \Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0); \vec{p}) = 0 \quad \text{for all } n + |w| \geq 0, \quad (39)$$

and for all A, B . Imposing these boundary conditions means that no regularizing counter terms for the corresponding operator product are introduced. The FE's for the CAG's with insertions may be integrated subject to these boundary conditions, and this will be our prescription for actually defining them. In the end, the cutoffs Λ, Λ_0 are taken away, and the limits will be controlled by the estimates that are given in the next section 3.

Regularized operator products for two or more insertions are denoted $\mathcal{L}_{n,l,D}^{\Lambda, \Lambda_0}(\otimes_i \mathcal{O}_{A_i}; \vec{p})$ and are defined for any $D \geq 0$. They are defined as the solutions to the FE (34), together with the boundary conditions

$$\partial^w \mathcal{L}_{n,l,D}^{0, \Lambda_0}(\otimes_i \mathcal{O}_{A_i}(x_i); \vec{0}) = 0 \quad \text{for } n + |w| \leq D, \quad (40)$$

$$\partial^w \mathcal{L}_{n,l,D}^{\Lambda_0, \Lambda_0}(\otimes_i \mathcal{O}_{A_i}(x_i); \vec{p}) = 0 \quad \text{for } n + |w| > D. \quad (41)$$

In particular, for $D = -1$, the normal products agree with the previously defined CAG's with multiple insertions, because they then satisfy the same boundary conditions and FE.

A useful property of the CAG's (both 'standard' and 'oversubtracted'), which follows from our choice of boundary conditions, is the following. Let \mathcal{O}_A be as usual a monomial in φ and its derivatives. Furthermore, for any multi-index $w \in \mathbb{N}^4$, let $\partial^w \mathcal{O}_A$ be the linear combination of monomials that are obtained by carrying out the derivatives in the obvious way. Then the CAG's are seen [7, 8] to satisfy the ‘Lowenstein rule’:

$$\begin{aligned} & \partial_{x_i}^w L_D^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(x_1) \otimes \cdots \otimes \mathcal{O}_{A_r}(0)) \\ = & \begin{cases} L_D^{\Lambda, \Lambda_0}(\mathcal{O}_{A_1}(x_1) \otimes \cdots \otimes \partial_{x_i}^w \mathcal{O}_{A_i}(x_i) \otimes \cdots \otimes \mathcal{O}_{A_r}(0)) & \text{for } r \geq 2, i \leq r-1, D \geq 0, \\ L_{D+|w|}^{\Lambda, \Lambda_0}(\partial_{x_1}^w \mathcal{O}_{A_1}(x_1)) & \text{for } r = 1, i = 1, D \geq [A_1]. \end{cases} \end{aligned} \quad (42)$$

This property is important in order to define insertions containing derivatives in a consistent way and has also been termed ‘action Ward identity’, or ‘Leibniz rule’. See [6, 5] for a discussion of such conditions in other setups of renormalization theory.

A major advantage of the CAG's for our purposes is that the OPE coefficients can be expressed in terms of them in relatively simple manner, as we now explain. For $F(\varphi)$ a differentiable functional of Schwartz space functions $\varphi \in \mathcal{S}(\mathbb{R}^4)$, let \mathcal{D}^A be the operator defined as

$$\mathcal{D}^A F = \frac{(-i)^{|w|}}{n! w!} \partial_{\vec{p}}^w \frac{\delta^n}{\delta \hat{\varphi}(p_1) \cdots \delta \hat{\varphi}(p_n)} F(\varphi) \Big|_{\hat{\varphi}=0, \vec{p}=0}, \quad \text{where } A = \{n, w\}. \quad (43)$$

Also, for a sufficiently smooth function f on \mathbb{R}^4 , let the Taylor expansion operator \mathbb{T}^j be defined as

$$\mathbb{T}^j f(x) = \sum_{|w|=j} \frac{x^w}{w!} \partial^w f(0). \quad (44)$$

Then the OPE coefficients are defined as follows:

Definition 2.1. For a finite UV-cutoff Λ_0 , the OPE coefficients, $\mathcal{C}_{AB}^C(x)$ are defined as follows:

1. Let $[C] - [A] - [B] < 0$. Then we define

$$\mathcal{C}_{AB}^C(x) := \mathcal{D}^C \left\{ \hbar L_{[C]-1}^{0, \Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0)) \right\}. \quad (45)$$

2. Let $[C] - [A] - [B] \geq 0$. Then we define

$$\begin{aligned} \mathcal{C}_{AB}^C(x) &:= \mathcal{D}^C \left\{ \hbar L_{[C]-1}^{0, \Lambda_0}((1 - \sum_{j=0}^{[C]-[A]-[B]-1} \mathbb{T}^j) \mathcal{O}_A(x) \otimes \mathcal{O}_B(0)) - \right. \\ &\quad \left. - L_{[C]-[B]}^{0, \Lambda_0}(\mathbb{T}^{[C]-[A]-[B]} \mathcal{O}_A(x)) L_{[B]}^{0, \Lambda_0}(\mathcal{O}_B(0)) \right\}. \end{aligned} \quad (46)$$

Our bounds in the subsequent sections, or those in [8], imply that we can remove the cutoff Λ_0 in the CAG's in the above formulas, and that the \mathcal{C}_{AB}^C are well-defined (as smooth functions for $x \in \mathbb{R}^4 \setminus \{0\}$) in the limit as $\Lambda_0 \rightarrow \infty$. The OPE coefficients of the theory without cutoffs are defined to be this limit.

For our analysis of the operator products we need the FE's expanded w.r.t. the number of fields and loops. For one insertion we obtain from (33):

$$\partial_\Lambda \mathcal{L}_{2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_A; p_1, \dots, p_{2n}) = \binom{2n+2}{2} \int_k \dot{C}^\Lambda(k) \mathcal{L}_{2n+2,l-1}^{\Lambda,\Lambda_0}(\mathcal{O}_A; k, -k, p_1, \dots, p_{2n}) \quad (47)$$

$$-4 \sum_{\substack{l_1+l_2=l \\ n_1+n_2=n+1}} n_1 n_2 \mathbb{S} \left[\mathcal{L}_{2n_1,l_1}^{\Lambda,\Lambda_0}(\mathcal{O}_A; q, p_1, \dots, p_{2n_1-1}) \dot{C}^\Lambda(q) \mathcal{L}_{2n_2,l_2}^{\Lambda,\Lambda_0}(p_{2n_1}, \dots, p_{2n}) \right]$$

with⁹ $q = p_{2n_1} + \dots + p_{2n}$.

When expanded out in moments and powers of \hbar the FE's for two insertions (34) read:

$$\partial_\Lambda \mathcal{L}_{2n,l,D}^{\Lambda,\Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B; p_1, \dots, p_{2n}) \quad (48)$$

$$= \binom{2n+2}{2} \int_k \dot{C}^\Lambda(k) \mathcal{L}_{2n+2,l-1,D}^{\Lambda,\Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B; k, -k, p_1, \dots, p_{2n})$$

$$-4 \sum_{\substack{l_1+l_2=l \\ n_1+n_2=n+1}} n_1 n_2 \mathbb{S} \left[\mathcal{L}_{2n_1,l_1,D}^{\Lambda,\Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B; q, p_1, \dots, p_{2n_1-1}) \dot{C}^\Lambda(q) \mathcal{L}_{2n_2,l_2}^{\Lambda,\Lambda_0}(p_{2n_1}, \dots, p_{2n}) \right. \\ \left. - \int_k \mathcal{L}_{2n_1,l_1}^{\Lambda,\Lambda_0}(\mathcal{O}_A; k, p_1, \dots, p_{2n_1-1}) \dot{C}^\Lambda(k) \mathcal{L}_{2n_2,l_2}^{\Lambda,\Lambda_0}(\mathcal{O}_B; -k, p_{2n_1}, \dots, p_{2n}) \right]$$

with $q = p_{2n_1} + \dots + p_{2n}$.

The symmetrization operator \mathbb{S} is defined as above in (22).

3 Bounds on CAG's

In this section, we will derive bounds on the CAG's, including those with insertions. These bounds will imply the existence of the limits $\Lambda \rightarrow 0$ and $\Lambda_0 \rightarrow \infty$, but they will also be sufficient to prove the main result Thm. 3 of this paper.

The bounds on the CAG's depend on the choice of the coupling constant g entering the flow equation via the boundary condition $\mathcal{L}_{4,0}^{0,\Lambda_0}(\vec{0}) = \frac{g}{4!}$. The loop expanded (inserted or

⁹Note that by symmetry and translation invariance $\mathcal{L}_{2n_2,l_2}^{\Lambda,\Lambda_0}(p_{2n_1}, \dots, p_{2n}) = \mathcal{L}_{2n_2,l_2}^{\Lambda,\Lambda_0}(-q, p_{2n_1}, \dots, p_{2n-1})$.

non inserted) CAG's depend on this coupling in an obvious way; the noninserted functions $\mathcal{L}_{2n,l}$ carry a power of $g^{\frac{2n-2}{2}+l}$ for example. To simplify the subsequent bounds, we will always set $g = 1$ in the following.

3.1 A collection of useful bounds

The following bounds which largely stem from [12] will be useful to control the solutions of the various FE's:

The Λ -derivative of the propagator (13) is given by

$$\dot{C}^\Lambda(p) = -\frac{2}{\Lambda^3} e^{-\frac{p^2+m^2}{\Lambda^2}} . \quad (49)$$

We find

1)

$$2\frac{m^3}{\Lambda^3} e^{-\frac{m^2}{\Lambda^2}} \leq 1 , \quad \frac{m^N}{\Lambda^N} e^{-\frac{m^2}{\Lambda^2}} \leq \sqrt{N!} . \quad (50)$$

2) For given momentum set (p_1, \dots, p_n) we use the (shorthand) definitions

$$\vec{p} \equiv (p_1, \dots, p_n) , \quad |\vec{p}|_n \equiv \sup_{J \subset \{1, \dots, n\}} \left| \sum_{i \in J} p_i \right| , \quad \vec{p}_{n+2} \equiv (\vec{p}, k, -k) . \quad (51)$$

Subsequently we sometimes simply write $|\vec{p}|$ instead of $|\vec{p}|_{2n}$. Then we claim

$$\int_{\frac{k}{\Lambda}} e^{-\frac{1}{2}(\frac{k}{\Lambda})^2} \log^\lambda(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa}, \frac{\kappa}{m})) \leq \log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})) + [\lambda!]^{1/2} , \quad \kappa = \sup(\Lambda, m) . \quad (52)$$

The proof is in [12] , Lemma 4 and (54)–(58).

3) For $s \in \mathbb{N}$

$$\begin{aligned} & \sum_{\lambda=0}^{\lambda=l-1} \frac{1}{2^\lambda \lambda!} \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{-s-1} \left(\log^\lambda(\sup(\frac{|\vec{p}|_{2n}}{\kappa'}, \frac{\kappa'}{m})) + [\lambda!]^{1/2} \right) \\ & \leq 5 \frac{\Lambda^{-s}}{s} \sum_{\lambda=0}^{\lambda=l-1} \frac{1}{2^\lambda \lambda!} \log^\lambda \sup(\frac{|\vec{p}|_{2n}}{\kappa}, \frac{\kappa}{m}) . \end{aligned} \quad (53)$$

We wrote $\kappa' = \sup(\Lambda', m)$. For the proof¹⁰ see Lemma 5 in [12].

¹⁰In fact, the proof in [12] is given for $\Lambda \geq m$, but it can be extended to $\Lambda < m$ without any problem.

4) For integers $n, n_1, n_2 \geq 1, l, l_1, \lambda_1, l_2, \lambda_2 \geq 0$

$$\sum_{\substack{l_1+l_2=l, \\ n_1+n_2=n+1, \\ \lambda_1 \leq l_1, \lambda_2 \leq l_2, \\ \lambda_1+\lambda_2=\lambda}} \frac{1}{(l_1+1)^2 (l_2+1)^2 n_1^2 n_2^2} \frac{n!}{n_1! n_2!} \frac{\lambda!}{\lambda_1! \lambda_2!} \frac{(n_1+l_1-1)! (n_2+l_2-1)!}{(n+l-1)!} \leq 20 \frac{1}{(l+1)^2} \frac{1}{n^2}. \quad (54)$$

For the proof see Lemma 2 in [12].

5) We will repeatedly use bounds on the Hermite polynomials $H_n(x) = (-1)^n e^{x^2} \frac{d}{dx^n} e^{-x^2}$:

$$H_n(x) \leq k \sqrt{n!} 2^{n/2} e^{x^2/2}, \quad k = 1.086 \dots \quad (55)$$

For a proof see [16], p. 324. It then follows directly from this bound that

$$|\partial^w e^{-\frac{q^2+m^2}{\Lambda^2}}| \leq k \Lambda^{-|w|} \sqrt{|w|!} 2^{\frac{|w|}{2}} e^{-\frac{q^2}{2\Lambda^2}} e^{-\frac{m^2}{\Lambda^2}}. \quad (56)$$

3.2 Bounds on higher derivatives of CAG's without insertions

Bounds on higher derivatives of CAG's are proven inductively with the aid of the flow equation. As compared to the bounds to be found in the literature [12] the new ingredient here is a sufficiently precise control of those bounds as regards their dependence on the number of derivatives $|w|$. We want to show

Proposition 3.1. There exists a constant $K > 0$ such that for $2n + |w| \geq 5$

$$|\partial^w \mathcal{L}_{2n,l}^{\Lambda, \Lambda_0}(p_1, \dots, p_{n-1})| \leq \sqrt{|w|!} \Lambda^{4-2n-|w|} K^{(2n+4l-4)(|w|+1)} (n+l-2)! \sum_{\lambda=0}^{\lambda=\ell(n,l)} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!}. \quad (57)$$

Here

$$\ell(n, l) = l \text{ if } n \geq 2, \quad \ell(n, l) = l - 1 \text{ if } n = 1. \quad (58)$$

The proposition is a consequence of the subsequent

Lemma 3.1. There exists a constant $K > 0$ such that for $2n + |w| \geq 5$

$$|\partial^w \mathcal{L}_{2n,l}^{\Lambda, \Lambda_0}(p_1, \dots, p_{2n-1})| \leq \sqrt{|w|!} \Lambda^{4-2n-|w|} \frac{K^{(2n+4l-4)(|w|+1)}}{(l+1)^2 n! n^3} (n+l-1)! \sum_{\lambda=0}^{\lambda=\ell} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!}. \quad (59)$$

Remark : The Lemma is sharper than the proposition, and the stated bound is suited as an inductive statement for its proof. Subsequently we will however use the (shorter) bound from the proposition.

Proof :

The proof is based on the standard inductive scheme which goes up in $n+l$ and for given $n+l$ goes up in l , and for given n, l descends in $|w|$. For $2n + |w| \leq 4$ we will use the bounds from the Theorem and Proposition¹¹ in [12] :

$$|\mathcal{L}_{4,l}^{\Lambda,\Lambda_0}(\vec{p})| \leq \frac{K^{2l}}{(l+1)^2 2^4} (1+l)! \sum_{\lambda=0}^{\lambda=l} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} , \quad (60)$$

$$|\partial^w \mathcal{L}_{2,l}^{\Lambda,\Lambda_0}(p)| \leq \sup(|p|, \kappa)^{2-|w|} \frac{K^{2l-1}}{(l+1)^2} l! \sum_{\lambda=0}^{\lambda=l-1} \frac{\log^\lambda(\sup(\frac{|p|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} . \quad (61)$$

A) *The first term on the r.h.s. of the FE*

Integrating the FE (23) w.r.t. the flow parameter Λ' from Λ to Λ_0 gives the following bound¹² for the first term on the r.h.s. of the FE (writing $\kappa' = \sup(\Lambda', m)$):

$$\begin{aligned} & \int_{\Lambda}^{\Lambda_0} d\Lambda' \int_k \frac{2}{\Lambda'^3} e^{-\frac{k^2+m^2}{\Lambda'^2}} \Lambda'^{4-(2n+2)-|w|} \sum_{\lambda=0}^{\lambda=l-1} \frac{\log^\lambda(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m}))}{2^\lambda \lambda!} \\ & \times \frac{(2n+1)(2n+2)}{2} \sqrt{|w|!} \frac{K^{(2n+4l-6)(|w|+1)}}{l^2 (n+1)! (n+1)^3} (n+l-1)! \\ & \leq \left(\frac{n}{n+1}\right)^3 (2n+1) \frac{K^{(2n+4l-6)(|w|+1)}}{l^2 n! n^3} (n+l-1)! \sqrt{|w|!} \sum_{\lambda=0}^{\lambda=l-1} \frac{1}{2^\lambda \lambda!} \\ & \times \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{3-2n-|w|} e^{-\frac{m^2}{\Lambda'^2}} \int_k e^{-\frac{k^2}{\Lambda'^2}} \log^\lambda(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m})) . \end{aligned} \quad (62)$$

Using

$$|\vec{p}|_{2n+2} \leq |\vec{p}| + |k| \quad (63)$$

we bound the momentum integral with the aid of (52). On performing the integral over Λ' in (62) and using (53) we therefore obtain the following bound for (62)

$$\Lambda^{4-2n-|w|} \frac{K^{(2n+4l-4)(|w|+1)}}{l^2 n! n^3} (n+l-1)! \sqrt{|w|!} \sum_{\lambda=0}^{\lambda=l} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} \quad (64)$$

¹¹We have slightly simplified the respective expressions which is possible if admitting for a slightly larger K as compared to [12].

¹²We assume $l \geq 1$, otherwise the contribution is zero.

$$\times [5 K^{-2(|w|+1)} 2^{|w|}] \left(\frac{n}{n+1}\right)^3 \frac{2n+1}{2n+|w|-4}.$$

We realize that (64) is smaller than the inductive bound divided by 2 if $K \geq 4$.

B) *The second term on the r.h.s. of the FE*

We assume without loss that $2n+4l \geq 6$, otherwise the contribution is zero. Subsequently we will also assume that neither term is a two-point function with $|w| \leq 1$. If one of them is so, we first have to bound the term $\sup(q, \kappa')^{2-|w|}$ arising from (61) together with the exponential $e^{-\frac{q^2}{2\Lambda'^2}}$ from the differentiated propagator, remember also (55, 56), by $2\Lambda'^{2-|w|}$. Afterwards this contribution can be absorbed into the subsequent proof at the cost of a factor of 2 in the lower bound on K .

Integrating the inductive bound on the *second* term on the r.h.s. of the FE from Λ to Λ_0 then gives us the following bound - where we also understand that the sup w.r.t. the permutations of the momentum attributions has been taken

$$\begin{aligned} & \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{8-(2n+2)-|w_1|-|w_2|} K^{(2n+4l-6)(|w_1|+|w_2|+2)} \sum_{\substack{l_1+l_2=l, \\ w_1+w_2+w_3=w, \\ n_1+n_2=n+1}} 2^{c_{\{w_i\}}} \frac{n_1}{(l_1+1)^2 n_1! n_1^3} \frac{n_2}{(l_2+1)^2 n_2! n_2^3} \\ & \times \sqrt{|w_1|!} (n_1+l_1-1)! \sum_{\lambda_1=0}^{\lambda_1=\ell_1} \frac{\log^{\lambda_1}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda_1} \lambda_1!} \frac{2}{\Lambda'^3} |\partial^{w_3} e^{-\frac{q^2+m^2}{\Lambda'^2}}| \\ & \times \sqrt{|w_2|!} (n_2+l_2-1)! \sum_{\lambda_2=0}^{\lambda_2=\ell_2} \frac{\log^{\lambda_2}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda_2} \lambda_2!} \\ & \leq \sum_{\substack{l_1+l_2=l, \\ n_1+n_2=n+1, \\ \lambda_1 \leq l_1, \lambda_2 \leq l_2}} \frac{1}{(l_1+1)^2 (l_2+1)^2} \frac{1}{n_1^2 n_2^2} \frac{n!}{n_1! n_2!} \frac{(\lambda_1+\lambda_2)!}{\lambda_1! \lambda_2!} \frac{(n_1+l_1-1)! (n_2+l_2-1)!}{(n+l-1)!} \\ & \times 2 K^{(2n+4l-6)(|w|+2)} \frac{(n+l-1)!}{n!} \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{6-2n-|w_1|-|w_2|} \frac{\log^{\lambda_1+\lambda_2}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda_1+\lambda_2} (\lambda_1+\lambda_2)!} \\ & \times \sum_{w_1+w_2+w_3=w} c_{\{w_i\}} \frac{2}{\Lambda'^3} |\partial^{w_3} e^{-\frac{q^2+m^2}{\Lambda'^2}}| \sqrt{|w_1|! |w_2|!}. \end{aligned}$$

Using (54, 56) we then arrive at the bound¹³

$$20 \frac{1}{(l+1)^2} \frac{1}{n^2} 2 K^{(2n+4l-6)(|w|+2)} \frac{1}{n!} (n+l-1)! \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{3-2n-|w|} \sum_{0 \leq \lambda \leq \ell} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda} \lambda!}$$

¹³note that if $2n = 2$ we have $2n_1 = 2n_2 = 2$, and the restriction to $\lambda \leq \ell$ in the sum over λ is justified.

$$\times \sum_{w_i} c_{\{w_i\}} 2^{\frac{1}{2}|w_3|} k \sqrt{|w_1|! |w_2|! |w_3|!} . \quad (65)$$

Using also (53) we verify the inductive bound

$$\Lambda^{4-2n-|w|} K^{(2n+4l-4)(|w|+1)} \frac{1}{(l+1)^2} \frac{1}{n^3} \frac{1}{n!} (n+l-1)! \sqrt{|w|!} \sum_{0 \leq \lambda \leq \ell} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!}$$

multiplied by $1/4$, on imposing the lower bound on K

$$K^{-2(|w|+2)} 40 \frac{n}{2n+|w|-4} \sum_{w_i} c_{\{w_i\}} 2^{\frac{1}{2}|w_3|} \leq 1/4 , \quad (66)$$

which is satisfied if

$$K \geq (640)^{\frac{1}{4}} 3^{\frac{1}{2}} .$$

□

The following variant of Lemma 3.1 is proven analogously :

Corollary 3.1. There exists a constant $K > 0$ such that for $2n + |w| \geq 5$

$$\begin{aligned} |\partial^w \mathcal{L}_{2n,l}^{\Lambda, \Lambda_0}(\vec{p})| &\leq \sqrt{|w|! (|w| + 2n - 4)!} \kappa^{4-2n-|w|} \frac{K^{(2n+4l-4)(|w|+1)}}{n!} (n+l-1)! \\ &\times \sum_{\lambda=0}^{\lambda=l} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} . \end{aligned} \quad (67)$$

As a consequence, the “gradient expansion” of the effective action

$$L^{0,\infty}(\varphi) = \sum_{n,l,w} \int_{\mathbb{R}^4} a_{w,n,l} \varphi(x) \partial^{w_1} \varphi(x) \cdots \partial^{w_{n-1}} \varphi(x) d^4x \quad (68)$$

with

$$a_{w,n,l} := \frac{\hbar^l}{w!} (-i\partial)^w \mathcal{L}_{n,l}^{0,\infty}(\vec{0}) \quad (69)$$

converges absolutely for each fixed loop order l , for each fixed n , and each Schwartz-space configuration φ such that $\hat{\varphi}(p)$ has support in a sufficiently small ball around $p = 0$ in momentum space. Furthermore, the expansion in l is locally Borel summable.

The bound (67) is weaker than the one of eq. (59), in the sense that it replaces $\sqrt{|w|!}$ by $\sqrt{|w|! (|w| + 2n - 4)!}$, and stronger in the sense that it replaces $\Lambda^{4-2n-|w|}$ by $\kappa^{4-2n-|w|}$ (14). In the proof there is no change as regards the first term on the r.h.s. of

the FE ; as regards the second term we use the bound (50)¹⁴ which permits to transform negative powers of Λ into negative powers of κ at the cost of a square root of a factorial. In the following we will only need Proposition 3.1.

3.3 Bounds on CAG's with one insertion

Throughout this section, we fix a monomial \mathcal{O}_A with $A = \{n', w'\}$, and we denote the dimension of this monomial by

$$D' := n' + |w'| = [A] . \quad (70)$$

For simplicity, we also assume that n' is even, the odd case can be treated similarly. We begin by rewriting the FE (47) with this insertion, with additional momentum derivatives:

$$\begin{aligned} \partial_\Lambda \partial^w \mathcal{L}_{2n,l,D}^{\Lambda,\Lambda_0}(\mathcal{O}_A; p_1, \dots, p_{2n}) &= \binom{2n+2}{2} \int_k \dot{C}^\Lambda(k) \partial^w \mathcal{L}_{2n+2,l-1,D}^{\Lambda,\Lambda_0}(\mathcal{O}_A; k, -k, p_1, \dots, p_{2n}) \\ &\quad - \sum_{\substack{l_1 + l_2 = l, \\ w_1 + w_2 + w_3 = w, \\ n_1 + n_2 = n+1}} 4n_1 n_2 c_{\{w_j\}} \mathbb{S} \left[\partial^{w_1} \mathcal{L}_{2n_1,l_1,D}^{\Lambda,\Lambda_0}(\mathcal{O}_A; q, p_1, \dots, p_{2n_1-1}) \partial^{w_3} \dot{C}^\Lambda(q) \partial^{w_2} \mathcal{L}_{2n_2,l_2}^{\Lambda,\Lambda_0}(p_{2n_1}, \dots, p_{2n}) \right] \end{aligned} \quad (71)$$

As always in this subsection, the insertion is at the point $x = 0$. Inspection of the FE shows that the renormalizability proof for the functions $\mathcal{L}_{n,l,D}^{\Lambda,\Lambda_0}$ can be performed on using the same inductive scheme as the one used for the $\mathcal{L}_{n,l}^{\Lambda,\Lambda_0}$, namely going up in $n + l$, for fixed $n + l$ ascending in l , and for fixed n, l descending in $|w|$. Bounds on the functions without insertions $\mathcal{L}_{n,l}^{\Lambda,\Lambda_0}$ are taken from the previous section. The boundary conditions for the $\mathcal{L}_{n,l,D}^{\Lambda,\Lambda_0}$ were given above in eqs. (37), (38). We consider the case $D = D' = n' + |w'|$, (36), and denote $\mathcal{L}_{2n,l,D}^{\Lambda,\Lambda_0}(\mathcal{O}_A; \vec{p})$ simply by $\mathcal{L}_{2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_A; \vec{p})$ if $D = [A]$.

Theorem 1. There exists a constant $K > 0$ such that for $\Lambda > 0$

$$\begin{aligned} |\partial^w \mathcal{L}_{2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_A; \vec{p})| &\leq \Lambda^{D-2n-|w|} K^{(4n+8l-4)|w|} K^{D(n+2l)^3} \\ &\times \sqrt{|w'|! |w|!} \sum_{\mu=0}^{d(D,n,l,w)} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^\mu \sum_{\lambda=0}^{\ell'(n,l)} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} . \end{aligned} \quad (72)$$

We set

$$d(D, n, l, w) := D(2n + 2l) + \sup(D + 1 - 2n - |w|, 0) , \quad (73)$$

$$\ell'(n, l) := 2l + n - 1 . \quad (74)$$

¹⁴Note that in the previous proof the factor of e^{-m^2/Λ^2} is simply bounded by one and thus is still at our disposal.

Proof :

We use the notation and the bounds of Proposition 3.1 and proceed similarly as there. If not written explicitly the arguments of ℓ' are supposed to be n, l , those of ℓ'_1 to be n_1, l_1 . We start considering

I) Irrelevant terms with $2n + |w| > D$:

A) The first term on the r.h.s. of the FE

Integrating the inductive bound on the *first* term from the r.h.s. of the FE (71) over Λ' between Λ_0 and Λ gives the following bound¹⁵:

$$\begin{aligned}
& \int_{\Lambda}^{\Lambda_0} d\Lambda' \int_k \frac{2}{\Lambda'^3} e^{-\frac{k^2+m^2}{\Lambda'^2}} \Lambda'^{D-(2n+2)-|w|} \sum_{\mu=0}^{d(D,n+1,l-1,w)} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|_{2n+2}}{\Lambda'} \right)^{\mu} \sum_{\lambda=0}^{\lambda=\ell'-1} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda} \lambda!} \\
& \quad \times \binom{2n+2}{2} \sqrt{|w'|! |w|!} K^{(4n+8l-8)|w|} K^{D(n+2l-1)^3} \\
& \leq \binom{2n+2}{2} \sqrt{|w'|! |w|!} K^{(4n+8l-8)|w|} K^{D(n+2l-1)^3} \sum_{\lambda=0}^{\lambda=\ell'-1} \frac{1}{2^{\lambda} \lambda!} \\
& \quad \times \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{D-1-2n-|w|} \sum_{\mu=0}^{d(D,n,l,w)} \frac{1}{\sqrt{\mu!}} \int_{k/\Lambda'} \left(\frac{|\vec{p}|_{2n+2}}{\Lambda'} \right)^{\mu} \log^{\lambda} \left(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m} \right) e^{-\frac{k^2}{\Lambda'^2}} . \quad (75)
\end{aligned}$$

Using (51) we show that

$$|\vec{p}|_{2n+2} \leq |\vec{p}| + |k|$$

and bound the momentum integral by

$$\begin{aligned}
& \sum_{\mu=0}^d \frac{1}{\sqrt{\mu!}} \sup_x \{ e^{-x^2/2} \left(\frac{|\vec{p}|_{2n+2}}{\Lambda'} \right)^{\mu} \} \int_x e^{-x^2/2} \log^{\lambda}(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m})) \quad (x = \frac{k}{\Lambda'}) \\
& \leq \left[\sum_{\mu=0}^d \frac{1}{\sqrt{\mu!}} \sum_{\rho=0}^{\mu} \binom{\mu}{\rho} \left(\frac{|\vec{p}|}{\Lambda'} \right)^{\rho} 2^{\frac{\mu-\rho}{2}} \left(\frac{\mu-\rho}{2} \right)! \right] \left[\log^{\lambda}(\sup(\frac{|\vec{p}|_{2n}}{\kappa'}, \frac{\kappa'}{m})) + (\lambda!)^{1/2} \right] \quad (76)
\end{aligned}$$

with the aid of (52). The first factor in (76) can then be bounded by

$$\begin{aligned}
& \sum_{\mu=0}^d \frac{1}{\sqrt{\mu!}} \sum_{\rho=0}^{\mu} \binom{\mu}{\rho} \left(\frac{|\vec{p}|}{\Lambda'} \right)^{\rho} \left(\frac{\mu-\rho}{2} \right)! 2^{\frac{\mu-\rho}{2}} \leq \sum_{\rho=0}^d \left(\frac{|\vec{p}|}{\Lambda'} \right)^{\rho} \sum_{\mu=\rho}^d \frac{1}{\sqrt{\mu!}} \binom{\mu}{\rho} \left(\frac{\mu-\rho}{2} \right)! 2^{\frac{\mu-\rho}{2}} \quad (77) \\
& \leq \sum_{\mu=0}^d \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda'} \right)^{\mu} \sum_{\rho=0}^{d-\mu} \binom{\rho+\mu}{\mu} \left(\frac{\rho}{2} \right)! 2^{\frac{\rho}{2}} \sqrt{\frac{\mu!}{(\rho+\mu)!}} \leq 2^d \sum_{\mu=0}^d \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda'} \right)^{\mu} ,
\end{aligned}$$

¹⁵Assuming $l \geq 1$, otherwise the contribution is zero, and writing $\kappa' = \sup(\Lambda', m)$.

where the last bound is obtained from Stirling and binomial type estimates.

Performing the integral over Λ' in (75) and using (53), we therefore obtain the following bound for (75):

$$\begin{aligned} & \Lambda^{D-2n-|w|} K^{D(n+2l)^3} K^{(4n+8l-4)|w|} \sum_{\mu=0}^d \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^\mu \sum_{\lambda=0}^{\lambda=\ell'-1} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} \\ & \times K^{-D[(n+2l)(n+2l-1)+1]-|w|} \left[5 \frac{(n+1)(2n+1)}{2n+|w|-D} 2^d K^{-2D(n+2l)(n+2l-1)-3|w|} \right]. \end{aligned} \quad (78)$$

As a consequence, this contribution to $\partial^w \mathcal{L}_{2n,l}^{\Lambda, \Lambda_0}(\mathcal{O}_A, \vec{p})$ **satisfies the inductive bound multiplied by**

$$\mathcal{A}(D, n, l, w) := 1/8 K^{-|w|} K^{-2D[(n+2l)(n+2l-1)+1]}, \quad (79)$$

if we assume K to be sufficiently large such that

$$5(n+1)(2n+1) 2^d K^{-2D(n+2l)(n+2l-1)-3|w|} \leq 1/8. \quad (80)$$

B) The second term on the r.h.s. of the FE

Integrating the inductive bound on the *second* term on the r.h.s. of the FE over Λ' between Λ and Λ_0 gives the following bound¹⁶, using that $|\vec{p}|_{2n_1}, |\vec{p}|_{2n_2} \leq |\vec{p}|_{2n} \equiv |\vec{p}|$ and taking the sup w.r.t. the permutations of the momentum assignments:

$$\begin{aligned} & \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{D+4-2n-2-|w_1|-|w_2|} K^{(4n_1+8l_1-4+2n_2+4l_2-4)(|w_1|+|w_2|+1)} K^{D(n_1+2l_1)^3} \\ & \times \sum_{\substack{l_1+l_2=l, \\ w_1+w_2+w_3=w, \\ n_1+n_2=n+1}} 4 c_{\{w_i\}} n_1 n_2 \sqrt{|w'|! |w_1|!} \sum_{\mu=0}^{d(D, n_1, l_1, w_1)} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda'}\right)^\mu \sum_{\lambda_1=0}^{\lambda_1=\ell'_1} \frac{\log^{\lambda_1}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda_1} \lambda_1!} \\ & \times \frac{2}{\Lambda'^3} |\partial^{w_3} e^{-\frac{q^2+m^2}{\Lambda'^2}}| \sqrt{|w_2|!} (n_2 + l_2 - 2)! \sum_{\lambda_2=0}^{\lambda_2=\ell'_2} \frac{\log^{\lambda_2}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda_2} \lambda_2!} \\ & \leq \sum_{\substack{l_1+l_2=l, \\ n_1+n_2=n+1, \\ \lambda_1 \leq \ell'_1, \lambda_2 \leq \ell'_2}} (n_2 + l_2)! 4 n_1 \frac{(\lambda_1 + \lambda_2)!}{\lambda_1! \lambda_2!} K^{(4n+8l-6)(|w|+1)} K^{D(n_1+2l_1)^3} \end{aligned}$$

¹⁶ Note that the lowest possible value of $4n + 8l - 4$ which may give a nonvanishing contribution on the r.h.s. is 4. This is realized for $(n = 2, l = 0)$. Thus the corresponding exponent of K in the inductive bound is never negative. A negative exponent could give a bound incompatible with the boundary contributions from (36).

$$\begin{aligned}
& \times \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{D+2-2n-|w_1|-|w_2|} \sum_{\mu=0}^{d(D,n_1,l_1,w_1)} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda'}\right)^{\mu} \frac{\log^{\lambda_1+\lambda_2}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda_1+\lambda_2} (\lambda_1 + \lambda_2)!} \\
& \times \sum_{w_1+w_2+w_3=w} c_{\{w_i\}} \frac{2}{\Lambda'^3} |\partial^{w_3}| e^{-\frac{q^2+m^2}{\Lambda'^2}} |w'|! |w_1|! |w_2|! .
\end{aligned}$$

Using (56) and the fact that $\ell'_1 + \ell'_2 \leq \ell'$, we arrive at the bound

$$\begin{aligned}
& \sum_{\substack{l_1+l_2=l, \\ n_1+n_2=n+1}} (n_2+l_2)! 4 n_1 \ell' 2^{\ell'} K^{(4n+8l-6)(|w|+1)} K^{D(n_1+2l_1)^3} \sum_{w_i} c_{\{w_i\}} 2^{\frac{1}{2}|w_3|} k \sqrt{|w'|! |w_3|! |w_1|! |w_2|!} \\
& \times \sum_{\mu=0}^{d(D,n_1,l_1,w_1)} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{\mu} \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{D-2n-|w|-1} \sum_{0 \leq \lambda \leq \ell'} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda} \lambda!} . \quad (81)
\end{aligned}$$

Using also (53) we verify the bound

$$\Lambda^{D-2n-|w|} K^{(4n+8l-4)|w|+D(n+2l)^3} \sqrt{|w'|! |w|!} \sum_{\nu=0}^{d(D,n_1,l_1,w_1)} \frac{1}{\sqrt{\nu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{\nu} \sum_{0 \leq \lambda \leq \ell'} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m}))}{2^{\lambda} \lambda!} , \quad (82)$$

multiplied by (79)–on imposing the lower bound on K

$$K^{-2D(n+2l)[(n+2l)-1]+(4n+8l-6)-|w|} 5k \sum_{\substack{l_1+l_2=l, \\ n_1+n_2=n+1}} (n_2+l_2)! 4 n_1 \ell' 2^{\ell'} \sum_{w_i} c_{\{w_i\}} 2^{\frac{|w_3|}{2}} \leq \frac{1}{8} \quad (83)$$

where we used that $n_1 + 2l_1 \leq n + 2l - 1$. Noting also $2n_1 + 2l_1 \leq 2n + 2l - 2$ we verify that

$$d(D, n_1, l_1, w_1) \leq d(D, n, l, w) , \quad (84)$$

with the aid of definition (73), so that (82) is bounded by (72), as required.

Adding the bounds on the first and second terms on the r.h.s. of the FE we verify the bound (72) multiplied by $2\mathcal{A}(D, n, l, w)$ for K sufficiently large to satisfy (80), (83).

II) Relevant terms at vanishing external momentum

Relevant terms - i.e. $2n + |w| \leq D$ - are first constructed at zero external momentum with the aid of the Taylor series

$$\partial_{\vec{p}}^v f_{2n}(\vec{p}) = \sum_{|w| \leq D-2n-|v|} \frac{\vec{p}^w}{w!} [\partial_{\vec{p}}^{w+v} f_{2n}](0) + \sum_{|w|=D+1-2n-|v|} \vec{p}^w \int_0^1 d\tau \frac{(1-\tau)^{|w|-1}}{(|w|-1)!} [\partial_{\vec{p}}^{w+v} f_{2n}](\tau \vec{p}) . \quad (85)$$

We note that for the relevant terms we also have to take into account the contribution from the boundary condition, see (36); the factor of $w! \delta_{w,w'}$ in the boundary condition

exhausts the factor of $\sqrt{w!w'!}$ present in the inductive bound (72), which thus cannot be sharpened in this respect. We consider the r.h.s. of the FE for the term $\partial_{\vec{p}}^{w+v} \mathcal{L}_{2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_A, \vec{0})$ with $2n + |w + v| \leq D$.

A) *The first term on the r.h.s. of the FE*

Integrating the FE (71) w.r.t. Λ' from 0 to Λ —assuming again without loss of generality $l \geq 1$ —gives the following bound for the first term on the r.h.s. of the FE:

$$\begin{aligned}
& \binom{2n+2}{2} K^{(4n+8l-8)|w+v|} K^{D(n+2l-1)^3} \sqrt{|w'|! |w+v|!} \int_0^\Lambda d\Lambda' \Lambda'^{D-(2n+2)-|w+v|} \frac{2}{\Lambda'^3} \\
& \quad \times \int_k e^{-\frac{k^2+m^2}{\Lambda'^2}} \sum_{\mu=0}^{d(D,n,l,w+v)} \frac{1}{\sqrt{\mu!}} \left(\frac{|k|}{\Lambda'}\right)^\mu \sum_{\lambda=0}^{\lambda=\ell'-1} \frac{\log^\lambda(\sup(\frac{|k|}{\kappa'}, \frac{\kappa'}{m}))}{2^\lambda \lambda!} \\
& \leq \binom{2n+2}{2} K^{(4n+8l-8)|w+v|} K^{D(n+2l-1)^3} \sqrt{|w'|! |w+v|!} \sum_{\mu=0}^{d(D,n,l,w+v)} \sum_{\lambda=0}^{\lambda=\ell'-1} \frac{1}{2^\lambda \lambda!} \\
& \quad \times 2 \int_0^\Lambda d\Lambda' \Lambda'^{D-(2n+1)-|w+v|} e^{-\frac{m^2}{\Lambda'^2}} \int_{\frac{|k|}{\Lambda'}} \frac{1}{\sqrt{\mu!}} \left(\frac{|k|}{\Lambda'}\right)^\mu \log^\lambda(\sup(\frac{|k|}{\kappa'}, \frac{\kappa'}{m})) e^{-\frac{k^2}{\Lambda'^2}} . \quad (86)
\end{aligned}$$

We bound the momentum integral as before in (76) by

$$2^{\frac{\mu}{2}} \frac{1}{\sqrt{\mu!}} \left(\frac{\mu}{2}\right)! [\log^\lambda(\frac{\kappa'}{m}) + (\lambda!)^{1/2}] . \quad (87)$$

Summing over μ , and using $(\frac{\mu}{2})! \leq 2^{-\mu/2} \sqrt{\mu+1} \sqrt{\mu!}$, the first term from (87) can then be bounded by

$$\sum_{\mu=0}^{d(D,n,l,w+v)} 2^{\frac{\mu}{2}} 2^{-\frac{\mu}{2}} \sqrt{\mu+1} \leq 2 d(D, n, l, w+v)^{3/2} . \quad (88)$$

Using that

$$\begin{aligned}
& \int_0^\Lambda d\Lambda' \Lambda'^{D-(2n+1)-|w+v|} \log^\lambda(\frac{\kappa'}{m}) e^{-\frac{m^2}{\Lambda'^2}} \\
& \leq \Lambda^{D-2n-|w+v|} \begin{cases} \log^\lambda(\frac{\kappa}{m}) & \text{if } D-2n-|w+v| > 0, \\ 2(\lambda+1)^{-1} \log^{\lambda+1}(\frac{\kappa}{m}) & \text{if } D-2n-|w+v| = 0, \end{cases} \quad (89)
\end{aligned}$$

we therefore obtain for (86) the bound

$$\begin{aligned}
& \binom{2n+2}{2} K^{(4n+8l-8)|w+v|} K^{D(n+2l-1)^3} \sqrt{|w'|! |w+v|!} \\
& \quad \times \Lambda^{D-2n-|w+v|} 2 d(D, n, l, w+v)^{3/2} 6 \sum_{\lambda=0}^{\lambda=\ell'} \frac{1}{2^\lambda \lambda!} \log^\lambda(\frac{\kappa}{m}) . \quad (90)
\end{aligned}$$

As a consequence, this contribution to $\partial^{w+v} \mathcal{L}_{2n,l}^{\Lambda, \Lambda_0}(\mathcal{O}_A, \vec{0})$ satisfies the inductive bound multiplied by (79) (with $w \rightarrow w+v$) under the condition that

$$12 \binom{2n+2}{2} d(D, n, l, w+v)^{3/2} K^{-2D(n+2l)(n+2l-1)-3|w+v|} \leq 1/8.$$

B) *The second term on the r.h.s. of the FE*

Integrating the inductive bound on the *second* term on the r.h.s. of the FE from 0 to Λ gives the following bound at zero momentum

$$\begin{aligned} & \sum_{\substack{l_1+l_2=l, \\ w_1+w_2+w_3=w+v, \\ n_1+n_2=n+1}} 4 n_1 n_2 \int_0^\Lambda d\Lambda' \Lambda'^{D+4-2n-2-|w_1|-|w_2|} K^{(4n_1+8l_1-4+2n_2+4l_2-4)(|w_1|+|w_2|+1)} K^{D(n_1+2l_1)^3} \\ & \times c_{\{w_i\}} \sqrt{|w'|! |w_1|!} \sum_{\lambda_1=0}^{\lambda_1=\ell'_1} \frac{\log^{\lambda_1}(\frac{\kappa'}{m})}{2^{\lambda_1} \lambda_1!} \frac{2}{\Lambda'^3} \left| \partial^{w_3} e^{-\frac{q^2+m^2}{\Lambda'^2}} \right|_{q=0} \sqrt{|w_2|!} (n_2+l_2-2)! \sum_{\lambda_2=0}^{\lambda_2=\ell'_2} \frac{\log^{\lambda_2}(\frac{\kappa'}{m})}{2^{\lambda_2} \lambda_2!} \\ & \leq \sum_{\substack{l_1+l_2=l, \\ n_1+n_2=n+1, \\ \lambda_1 \leq \ell'_1, \lambda_2 \leq \ell'_2}} (n_2+l_2)! 4 n_1 \frac{(\lambda_1+\lambda_2)!}{\lambda_1! \lambda_2!} K^{(4n+8l-6)(|w|+|v|+1)} K^{D(n_1+2l_1)^3} \sum_{\substack{w_1+w_2+w_3 \\ =w+v}} c_{\{w_i\}} \\ & \times \int_0^\Lambda d\Lambda' \Lambda'^{D+2-2n-|w_1|-|w_2|} \frac{\log^{\lambda_1+\lambda_2}(\frac{\kappa'}{m})}{2^{\lambda_1+\lambda_2} (\lambda_1+\lambda_2)!} \frac{2}{\Lambda'^3} \left| \partial^{w_3} e^{-\frac{q^2+m^2}{\Lambda'^2}} \right|_{q=0} \sqrt{|w'|! |w_1|! |w_2|!}, \\ & \text{remembering (74) and (58) which imply that } \ell'_1 + \ell'_2 \leq \ell'. \text{ Using (56) we obtain the bound} \\ & \sum_{\substack{l_1+l_2=l, \\ n_1+n_2=n+1}} (n_2+l_2)! 4 n_1 l_2^l K^{(4n+8l-6)(|w+v|+1)} K^{D(n_1+2l_1)^3} \sum_{w_i} c_{\{w_i\}} k 2^{\frac{1}{2}|w_3|} \sqrt{|w'|! |w_3|! |w_1|! |w_2|!} \\ & \times \int_0^\Lambda d\Lambda' \Lambda'^{D-2n-|w+v|-1} e^{-\frac{m^2}{\Lambda'^2}} \sum_{0 \leq \lambda \leq \ell'} \frac{\log^\lambda(\frac{\kappa'}{m})}{2^\lambda \lambda!}. \end{aligned} \quad (91)$$

Using also (53) and proceeding as in (89, 90) we verify the inductive bound (72)

$$\Lambda^{D-2n-|w+v|} K^{(4n+8l-4)|w+v|+D(n+2l)^3} \sqrt{|w'|! |w+v|!} \sum_{0 \leq \lambda \leq \ell'} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda \lambda!},$$

multiplied by (79) (with $w \rightarrow w+v$) on imposing the lower bound on K

$$6 K^{-2D(n+2l)(n+2l-1)+(4n+8l-6)-|w+v|} 5k \sum_{\substack{l_1+l_2=l, \\ n_1+n_2=n+1}} (n_2+l_2)! 4 n_1 \ell' 2^{\ell'} \sum_{\sum w_i=w+v} c_{\{w_i\}} 2^{\frac{|w_3|}{2}} \leq \frac{1}{8}. \quad (92)$$

For $|w + v| + 2n = D$ we have to add the boundary term from (36). Since it is non-vanishing only if $w + v = w'$ we can bound it by $\sqrt{|w + v|!|w'|!}$ and it is thus accommodated for by the bound from Theorem 1 multiplied by $\frac{1}{8}$, remember also footnote 16.

III) *Schwinger functions with $|v| + 2n \leq D$ at arbitrary external momenta*

We have to sum the series (85).

$$\begin{aligned}
& |\partial_{\vec{p}}^v \mathcal{L}_{2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_A, \vec{p})| = \\
& \left| \sum_{|w| \leq D-2n-|v|} \frac{\vec{p}^w}{w!} \partial_{\vec{p}}^{w+v} \mathcal{L}_{2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_A, \vec{0}) + \sum_{|w|=D+1-2n-|v|} \vec{p}^w \int_0^1 \frac{(1-\tau)^{|w|-1}}{(|w|-1)!} \partial_{\vec{p}}^{w+v} \mathcal{L}_{1,2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_A, \tau \vec{p}) \right| \\
& \leq \left[\sum_{|w| \leq D-2n-|v|} \left(\frac{|\vec{p}|}{\Lambda}\right)^{|w|} 4 \mathcal{A}(D, n, l, w+v) K^{(4n+8l-4)|w+v|} \frac{\sqrt{|w'|!|w+v|!}}{w!} \sum_{\lambda=0}^{\lambda=\ell'} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda \lambda!} \right. \\
& \quad \left. + \sum_{|w|=D+1-2n-|v|} 4 \mathcal{A}(D, n, l, w+v) \left(\frac{|\vec{p}|}{\Lambda}\right)^{|w|} |w| K^{(4n+8l-4)|w+v|} \frac{\sqrt{|w'|!|w+v|!}}{|w|!} \int_0^1 d\tau (1-\tau)^{|w|-1} \right. \\
& \quad \left. \times \sum_{\mu=0}^{d(D,n,l,w+v)} \frac{1}{\sqrt{\mu!}} \left(\frac{\tau|\vec{p}|}{\Lambda}\right)^\mu \sum_{\lambda=0}^{\lambda=\ell'} \frac{\log^\lambda(\sup(\frac{\tau|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} \right] K^{D(n+2l)^3} \Lambda^{D-2n-|v|}.
\end{aligned} \tag{93}$$

where \mathcal{A} is given in (79). We used the induction hypothesis, after transforming powers of p into powers of p over Λ multiplied by powers of Λ .

Using the following estimate

$$\frac{\sqrt{|w'|!|w+v|!}}{w!} \leq \frac{\sqrt{|w'|!|v|!}}{\sqrt{|w|!}} 2^{\frac{|w+v|}{2}} \frac{|w|!}{w!} \leq \frac{\sqrt{|w'|!|v|!}}{\sqrt{|w|!}} 2^{\frac{|w|+|v|}{2}} (8n)^{|w|},$$

we obtain a bound for $2n + |v| \leq D$:

$$\begin{aligned}
& |\partial_{\vec{p}}^v \mathcal{L}_{2n,l}^{\Lambda,\Lambda_0}(\mathcal{O}_A, \vec{p})| \leq \Lambda^{D-2n-|v|} \sqrt{|w'|!|v|!} K^{D(n+2l)^3} \\
& \times \left[\sum_{w, 2n+|v+w| \leq D} 4 \mathcal{A}(D, n, l, w+v) \frac{1}{\sqrt{|w|!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{|w|} 2^{\frac{|w|+|v|}{2}} (8n)^{|w|} K^{(4n+8l-4)|w+v|} \sum_{\lambda=0}^{\lambda=\ell'} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda \lambda!} \right. \\
& \quad \left. + \sum_{w, |w|=D+1-2n-|v|} 4 \mathcal{A}(D, n, l, w+v) \frac{1}{\sqrt{|w|!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{|w|} |w| K^{(4n+8l-4)|w+v|} 2^{\frac{|w|+|v|+2}{2}} (8n)^{|w|} \right. \\
& \quad \left. \times \int_0^1 d\tau (1-\tau)^{|w|-1} \sum_{\mu=0}^{d(D,n,l,w+v)} \frac{1}{\sqrt{\mu!}} \left(\frac{\tau|\vec{p}|}{\Lambda}\right)^\mu \sum_{\lambda=0}^{\lambda=\ell'} \frac{\log^\lambda(\sup(\frac{\tau|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} \right]
\end{aligned} \tag{95}$$

$$\begin{aligned}
&\leq \Lambda^{D-2n-|v|} \sqrt{|w'|! |v|!} K^{D(n+l)^3} \\
&\times \left[\sum_{|w, 2n+|v+w|\leq D} 4 \mathcal{A}(D, n, l, w) \frac{1}{\sqrt{|w|!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^{|w|} 2^{\frac{|w|+|v|}{2}} (8n)^{|w|} K^{(4+8l-4)|w+v|} \sum_{\lambda=0}^{\lambda=\ell'} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda \lambda!} \right. \\
&\quad + \sum_{|w|=D+1-2n-|v|} 4 \mathcal{A}(D, n, l, w) |w| K^{(4n+8l-4)|w+v|} 2^{\frac{|w|+|v|+2}{2}} (8n)^{|w|} \\
&\quad \times \left. \sum_{\mu=0}^{d(D, n, l, w+v)+|w|} \frac{2^{\frac{\mu}{2}}}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^\mu \sum_{\lambda=0}^{\lambda=\ell'} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} \right]. \tag{96}
\end{aligned}$$

These bounds are compatible with the induction hypothesis since

i)

$$d(D, n, l, w+v) + |w| \leq d(D, n, l, v) \tag{97}$$

for $|w| \leq D - 2n - |v| + 1$, as a consequence of the definition of d (73),

ii)

$$\sum_{|w|\leq D+1-2n-|v|} (8n)^{|w|} 2^{\frac{d(D, n, l, v)+|w|}{2}} |w| K^{(4n+8l-4)|w|} 4 \mathcal{A}(D, n, l, w) \leq 1$$

for K sufficiently large.

□

Corollary 3.2. For $\Lambda \leq m$ and K sufficiently large we have the bounds

$$\begin{aligned}
&|\partial^w \mathcal{L}_{2n, l}^{\Lambda, \Lambda_0}(\mathcal{O}_A, \vec{p})| \leq m^{D-2n-|w|} K^{(4n+8l-4)|w|} K^{D(n+2l)^3} \\
&\times \sqrt{|w'|! |w|! [2n + |w| - D]_+!} \sum_{\mu=0}^{d(D, n, l, w)} \left(\frac{|\vec{p}|}{m}\right)^\mu \sum_{\lambda=0}^{\lambda=\ell'} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda \lambda!}. \tag{98}
\end{aligned}$$

Remark: These bounds show that the functions $\mathcal{L}_{2n, l}^{\Lambda, \Lambda_0}(\mathcal{O}_A; \vec{p})$ have a convergent Taylor expansion around zero momentum, since the growth of the Taylor coefficients is bounded by $\tilde{K}^{|w|} |w|!$ for w large and suitable \tilde{K} .

Proof: To prove the Corollary we may insert the bounds of Theorem 1 on the r.h.s. of the FE. We may then use the factors $\exp(-m^2/\Lambda'^2)$ present in both terms to bound the negative powers of Λ by a square root of a factorial :

$$\int_0^m d\Lambda' \exp(-m^2/\Lambda'^2) \frac{\Lambda'^{D-2n-|w|-\mu-1}}{\sqrt{\mu!}} \leq m^{D-2n-|w|-\mu} \frac{\sqrt{(2n + |w| + \mu - D)_+!}}{\sqrt{\mu!}}$$

$$\leq 2^{\frac{1}{2}(2n+|w|+\mu-D)_+} m^{D-2n-|w|-\mu} \sqrt{(2n+|w|-D)_+!}.$$

These bounds cannot serve as a viable induction hypothesis however, since the powers of momenta (now without $\frac{1}{\sqrt{\mu!}}$) would create additional square roots of factorials in the next step of induction. \square

For later use, we also note the

Corollary 3.3. The inductive proof of Theorem 1 is valid for a somewhat larger constant K also if we replace d in the statement of the theorem by $2d$, or if we replace ℓ' by $\ell' + 1$.

Proof: The key properties for d that enter the proof are the inequalities (84) and (97), which are evidently also satisfied for $2d$. The key properties required for $\ell' = \ell'(n, l)$ are that $\ell'(n+1, l-1) < \ell'(n, l)$ for $l \geq 1$ and $\ell'(n_1, l_1) + \ell(n_2, l_2) < \ell'(n, l)$ on the r.h.s. of the FE, where $\ell(n, l)$ is as in eq. (58). These properties are evidently also satisfied by the quantity $\ell' + 1$. \square

3.4 Bounds on normal products with two insertions

We now provide bounds on the normal products $\mathcal{L}_{n,l,D}^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B; \vec{p})$ with two insertions. Each of these insertions is a monomial in the basic fields with $A = \{n', w'\}$ and $B = \{n'', w''\}$. Again, we will assume for simplicity that both n' and n'' are even. We will use the notation D' for the combined dimension of the two insertions,

$$D' := [A] + [B] = n' + n'' + |w'| + |w''|. \quad (99)$$

These normal products were defined above in sec. 2 as the solutions to the FE's (48), and the boundary conditions are given above in eq. (40) and eq. (41). Our bounds are given in the following theorem:

Theorem 2. There exists a constant $K > 0$ such that for $|w| \leq D' + 1$:

$$|\partial^w \mathcal{L}_{2n,l,D'}^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B; \vec{p})| \leq \Lambda^{D'-2n-|w|} K^{(4n+8l-4)|w|} K^{D'(n+2l)^3} \sqrt{|w|! |w'|! |w''|!} \sum_{\mu=0}^{d'(n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda}\right)^\mu \sum_{\lambda=0}^{\lambda=\ell'+1} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!}, \quad (100)$$

$$d' = 2[D'(2n+2l) + \sup(D'+1-2n-|w|, 0)], \quad \ell'(n, l) = 2l + n - 1. \quad (101)$$

Proof:

To prove this theorem, we use the FE's for normal products with two insertions, given above in eq. (48). We apply a derivative $\partial_{\vec{p}}^w$ to both sides of the equation. Then we integrate the FE's over Λ subject to the appropriate boundary condition, using the same inductive scheme as described in the previous subsections. Depending on the boundary condition, we again have to distinguish the cases $2n + |w| \leq D'$ and $2n + |w| > D'$. The right side of the FE, eq. (48), has three terms. The first two terms involve the CAG's with two insertions, whereas the last term only involves the CAG's with one insertion, for which we already have the bounds in Theorem 1. The structure of the bound for the CAG's with two insertions claimed in the theorem is exactly the same as that for one insertion, and the first two terms on the right side of the FE also have exactly the same structure as the corresponding two terms in the FE for one insertion. Therefore, in view of Cor. 3.3, the first two terms in the FE can be treated in literally the same manner as in the previous section with $D = D'$ there. The third term on the right side of the FE has a different form, but it involves only the CAG's with one insertion, for which we already have bounds. Thus, we can concentrate only on the third term on the right side of the FE, and we need to show that this term satisfies our inductive bound. We begin with the following

Lemma 3.2. For $|w| \leq D' + 1$, $n + 1 = n_1 + n_2$, $l = l_1 + l_2$, we have the following bound:

$$\begin{aligned} & \left| \partial_{\vec{p}}^w \int_k \mathcal{L}_{2n_1, l_1}^{\Lambda, \Lambda_0}(\mathcal{O}_A; k, p_1, \dots, p_{2n_1-1}) \dot{C}^\Lambda(k) \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(\mathcal{O}_B; -k, p_{2n_1}, \dots, p_{2n}) \right| \\ & \leq M K_1^{(4n+8l-4)|w|} K_1^{D'(n+2l)^3} \sqrt{|w|! |w'|! |w''|!} \Lambda^{D'-2n-|w|-1} e^{-m^2/\Lambda^2} \\ & \quad \times \sum_{\mu=0}^{d'(n, l, w, D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda} \right)^\mu \sum_{\lambda=0}^{\lambda=\ell'} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})) + \sqrt{\lambda!}}{2^\lambda \lambda!}. \end{aligned} \quad (102)$$

Here K_1 is the constant from Theorem 1, and $M = 5^{|w|} 2^{|w|/2} 2^{2d'} (\ell' + 1) 2^{\ell'+1}$.

Proof: We can pull the $\partial_{\vec{p}}^w$ inside the integral. Then we first use the transformation properties of the CAG's under translations to write

$$\begin{aligned} & \partial_{\vec{p}}^w [\mathcal{L}_{2n_1, l_1}^{\Lambda, \Lambda_0}(\mathcal{O}_A(x); k, p_1, \dots, p_{2n_1-1}) \dot{C}^\Lambda(k) \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(\mathcal{O}_B(0); -k, p_{2n_1}, \dots, p_{2n})] \\ & = \sum_{w_1+w_2+w_3=w} c_{\{w_i\}} \partial_{\vec{p}}^{w_3} e^{ix(k+p_1+\dots+p_{2n_1-1})} \partial_{\vec{p}}^{w_1} \mathcal{L}_{2n_1+1, l_1}^{\Lambda, \Lambda_0}(\mathcal{O}_A(0); k, p_1, \dots, p_{2n_1-1}) \\ & \quad \times \dot{C}^\Lambda(k) \partial_{\vec{p}}^{w_2} \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(\mathcal{O}_B(0); -k, p_{2n_1}, \dots, p_{2n}) \end{aligned} \quad (103)$$

Now, the \vec{p} -derivatives on $\partial_{\vec{p}}^{w_3} e^{ix(k+p_1+\dots+p_{2n_1-1})}$ can be converted into k -derivatives, and then in the subsequent k -integral in (102) moved onto the other terms by means of a

partial integration, because the integrand decays sufficiently rapidly for large k by the bounds in Theorem 1. We can then insert these bounds, and we can also use the standard multi-nomial bound $c_{\{w_i\}} \leq 3^{|w|}$. Carrying out these manipulations, and using also (56), and the inequality $D'(n+2l)^3 \geq [A](n_1+2l_1)^3 + [B](n_2+2l_2)^3$ in view of (99), results in the bound:

$$\begin{aligned} & \left| \partial_{\vec{p}}^w \int_k \mathcal{L}_{2n_1, l_1}^{\Lambda, \Lambda_0}(\mathcal{O}_A; k, p_1, \dots, p_{2n_1-1}) \dot{C}^\Lambda(k) \mathcal{L}_{2n_2, l_2}^{\Lambda, \Lambda_0}(\mathcal{O}_B; -k, p_{2n_1}, \dots, p_{2n}) \right| \\ & \leq M_0 K_1^{(4n+8l-4)|w|} K_1^{D'(n+2l)^3} \sqrt{|w|! |w'|! |w''|!} \Lambda^{D'-2n-|w|-5} e^{-m^2/\Lambda^2} \\ & \quad \times \sum_{\mu=0}^{d_1+d_2} \frac{1}{\sqrt{\mu!}} \int_k e^{-|k|^2/2\Lambda^2} \left(\frac{|\vec{p}| + |k|}{\Lambda} \right)^\mu \sum_{\lambda=0}^{\ell'} \frac{\log^\lambda(\sup(\frac{|\vec{p}|+|k|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!}, \end{aligned} \quad (104)$$

for the constant K_1 provided by Theorem 1, and $M_0 = 5^{|w|} 2^{\ell'+1} 2^{d_1+d_2} (\ell'+1) 2^{|w|/2}$. Here, $n+1 = n_1+n_2$, $l = l_1+l_2$, and $d_1 = [A](2n_1+2l_1) + \sup([A]+1-2n_1-|w_1|, 0)$, $d_2 = [B](2n_2+2l_2) + \sup([B]+1-2n_2-|w_2|, 0)$ is as in Theorem 1. Using that $|w| \leq D'+1$, we can show that $d_1+d_2 \leq d'$, so we can replace the upper limit of the sum over μ by d' . Furthermore, we can now bound the k -integral in (104) exactly as in (76) and (77), leading to the bound

$$\begin{aligned} & \sum_{\mu=0}^{d'} \int_{k/\Lambda} e^{-|k|^2/2\Lambda^2} \left(\frac{|\vec{p}| + |k|}{\Lambda} \right)^\mu \sum_{\lambda=0}^{\ell'} \frac{\log^\lambda(\sup(\frac{|\vec{p}|+|k|}{\kappa}, \frac{\kappa}{m}))}{2^\lambda \lambda!} \\ & \leq 2^{d'} \sum_{\mu=0}^{d'(n,l,w,D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda} \right)^\mu \sum_{\lambda=0}^{\ell'} \frac{\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})) + \sqrt{\lambda!}}{2^\lambda \lambda!}. \end{aligned} \quad (105)$$

Inserting this into the previous bound gives the statement of the lemma. \square

We now return to the inductive step, which consists in integrating $\partial_{\vec{p}}^w$ on the right side of the FE eq. (48) against Λ , subject to appropriate boundary conditions. Concerning these boundary conditions, we must as usual consider separately two cases:

The case $2n+|w| > D'$: In this case the boundary condition is $\partial^w \mathcal{L}_{2n,l,D'}^{\Lambda_0, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B; \vec{p}) = 0$, so we integrate the right side of the FE differentiated by $\partial_{\vec{p}}^w$ from Λ_0 down to Λ . We have to consider the three terms on the right side separately. The first two can be handled as in the previous subsection. So we need to focus only on the third term on the right side

of the FE. Using the previous lemma, and the inequality (53), we get

$$\begin{aligned}
& \left| \int_{\Lambda}^{\Lambda_0} d\Lambda' \partial_{\vec{p}}^w \int_k \mathcal{L}_{2n_1, l_1}^{\Lambda', \Lambda_0}(\mathcal{O}_A; k, p_1, \dots, p_{2n_1-1}) \dot{C}^{\Lambda'}(k) \mathcal{L}_{2n_2, l_2}^{\Lambda', \Lambda_0}(\mathcal{O}_B; -k, p_{2n_1}, \dots, p_{2n}) \right| \\
& \leq M K_1^{(4n+8l-4)|w|} K_1^{D'(n+2l)^3} \sqrt{|w|! |w'|! |w''|!} \int_{\Lambda}^{\Lambda_0} d\Lambda' \Lambda'^{-1+D'-2n-|w|} e^{-m^2/\Lambda'^2} \\
& \quad \times \sum_{\mu=0}^{d'(n, l, w, D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda'} \right)^{\mu} \sum_{\lambda=0}^{\ell'} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m})) + \sqrt{\lambda!}}{2^{\lambda} \lambda!} \\
& \leq 10M K_1^{(4n+8l-4)|w|} K_1^{D'(n+2l)^3} \sqrt{|w|! |w'|! |w''|!} \Lambda^{D'-2n-|w|} \\
& \quad \times \sum_{\mu=0}^{d'(n, l, w, D')} \frac{1}{\sqrt{\mu!}} \left(\frac{|\vec{p}|}{\Lambda} \right)^{\mu} \sum_{\lambda=0}^{\ell'} \frac{\log^{\lambda}(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))}{2^{\lambda} \lambda!} . \tag{106}
\end{aligned}$$

In order to bound the Λ' -integral of the third term on the right side of the FE, we must additionally multiply this by $4n_1n_2$, apply the symmetrization operator \mathbb{S} , and sum over n_1, n_2, l_1, l_2 , subject to $n+1 = n_1 + n_2, l = l_1 + l_2$. Then we see that we reproduce the inductive bound in Theorem 2 on the FE multiplied by $1/8$ by choosing $K \geq K_1$ sufficiently large such that

$$40(l+1)(n+1)^3 M K_1^{D'(n+l)^3} \leq \frac{1}{8} K^{D'(n+l)^3}, \tag{107}$$

which is possible in view of $D' \geq 2n + |w| + 1$.

The case $2n+|w| \leq D'$: In this case the boundary condition is $\partial^w \mathcal{L}_{2n, l, D'}^{0, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B; \vec{0}) = 0$, so we integrate the right side of the FE differentiated by $\partial_{\vec{p}}^w$ from Λ down to 0. We have to consider the three terms on the right side separately. The first two can be handled as in the previous subsection. So we need to focus again only on the third term on the right side of the FE. This is done first for zero momentum $\vec{p} = \vec{0}$, and the results for arbitrary momentum are then constructed using the Taylor formula with remainder. Using the previous lemma, we now get

$$\begin{aligned}
& \left| \int_0^{\Lambda} d\Lambda' \partial_{\vec{p}}^w \int_k \mathcal{L}_{2n_1, l_1}^{\Lambda', \Lambda_0}(\mathcal{O}_A; k, 0, \dots, 0) \dot{C}^{\Lambda'}(k) \mathcal{L}_{2n_2, l_2}^{\Lambda', \Lambda_0}(\mathcal{O}_B; -k, 0, \dots, 0) \right| \\
& \leq M K_1^{(4n+8l-4)|w|} K_1^{D'(n+2l)^3} \sqrt{|w|! |w'|! |w''|!} \\
& \quad \times \int_0^{\Lambda} d\Lambda' \Lambda'^{D'-2n-|w|-1} e^{-m^2/\Lambda'^2} \sum_{\lambda=0}^{\ell'} \frac{\log^{\lambda}(\frac{\kappa'}{m}) + \sqrt{\lambda!}}{2^{\lambda} \lambda!}, \tag{108}
\end{aligned}$$

noting that there is no boundary term. The Λ' -integral is now bounded by

$$\begin{aligned}
& \int_0^\Lambda d\Lambda' \Lambda'^{D'-2n-|w|-1} e^{-m^2/\Lambda'^2} \sum_{\lambda=0}^{\ell'} \frac{\log^\lambda(\frac{\kappa'}{m}) + \sqrt{\lambda!}}{2^\lambda \lambda!} \\
& \leq \Lambda^{D'-2n-|w|} \sum_{\lambda=0}^{\ell'} \frac{1}{2^\lambda \lambda!} \begin{cases} \frac{1}{\lambda+1} \log^{\lambda+1}(\frac{\kappa}{m}) + \sqrt{\lambda!} \log(\frac{\kappa}{m}) + 1 & \text{if } D' - 2n - |w| = 0, \\ \frac{\log^\lambda(\frac{\kappa}{m}) + \sqrt{\lambda!}}{D' - 2n - |w|} & \text{if } D' - 2n - |w| > 0 \end{cases} \\
& \leq 6(l+1) \Lambda^{D'-2n-|w|} \sum_{\lambda=0}^{\ell'+1} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda \lambda!}. \tag{109}
\end{aligned}$$

In order to bound the Λ' integral of the third term on the right side of the FE, we must additionally multiply (108) by $4n_1n_2$, apply the symmetrization operator \mathbb{S} , and sum over n_1, n_2, l_1, l_2 , subject to $n+1 = n_1 + n_2, l = l_1 + l_2$. Then we see that we reproduce the inductive bound in Theorem 2 on the FE multiplied by $1/8$ provided that

$$4 \cdot 6(l+1)(\ell'+1)(n+1)^3 M K_1^{D'(n+l)^3} \leq \frac{1}{8} K^{D'(n+l)^3}, \quad K_1 \leq K \tag{110}$$

which can be satisfied for K sufficiently large in view of $|w| \leq D' + 1$. The bounds at non-zero momentum are obtained using the Taylor expansion with remainder technique as in eq. (93), but now for two insertions. The arguments are in parallel with the case of one insertion, noting that d' satisfies the key property $d'(D', n, l, w+v) + |w| \leq d'(D', n, l, v)$ analogous to (97). Thus, each of the two terms in eq. (93) satisfies the inductive bound multiplied by $1/8$ for sufficiently large K .

Hence, we have seen that $\partial_{\vec{p}}^w$ of the third term in the FE (48), integrated against Λ , can be estimated by $1/2$ of the inductive bound. The first two terms can be treated in the same manner as the corresponding terms in the FE with one insertion, and can thereby be bounded by $1/2$ times the inductive bound as well for sufficiently large K . This concludes the proof of the theorem. \square

We can insert the bound obtained in the previous theorem one more time into the FE's and integrate from 0 to m . If this is done, and if we also carry out the sum over μ in the bound, we obtain, in the same way as Corollary 3.2 was obtained¹⁷ from Theorem 1:

Corollary 3.4. There exists a constant $K > 0$ such that:

$$|\mathcal{L}_{2n,l,D'}^{0,\Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0); \vec{p})| \leq m^{D'-2n} K^{D'(n+2l)^3} \sqrt{|w'|! |w''|!} \sup(1, \frac{|\vec{p}|}{m})^{d'} \sum_{\lambda=0}^{2l+n} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda \lambda!}. \tag{111}$$

¹⁷The factor $\sqrt{(2n-D')_+!}$ can be absorbed by choosing K slightly larger.

Here, $d' = 2[D'(2n + 2l) + \sup(D' + 1 - 2n, 0)]$.

4 Bound on the remainder in the OPE

The bounds on the CAG's obtained in the previous section put us in a position to give the proof of our main result, namely the bound on the remainder of the OPE. As stated in the introduction, we introduce test-functions f_{p_i} chosen such that the support $\text{supp } \hat{f}_{p_i} \subset \{q \in \mathbb{R}^4 \mid |p_i - q| \leq \epsilon\}$ for some arbitrary but fixed ϵ . In terms of these test-functions, the spectator fields are defined as $\varphi(f_{p_i}) = \int d^4x \varphi(x) f_{p_i}(x)$. We use the same notation as in the previous section concerning the composite fields: $D' = [A] + [B]$, and $A = \{n', w'\}, B = \{n'', w''\}$. Our result, which we presented already in the introduction, is

Theorem 3. Let the sum \sum_C in the operator product expansion (1) be over all C such that

$$[C] - [A] - [B] \leq \Delta \quad (112)$$

where Δ is some positive integer. Then for each such Δ , we have the following bound for the “remainder” in the OPE *in loop order l* :

$$\begin{aligned} & \left| \left\langle \mathcal{O}_A(x) \mathcal{O}_B(0) \varphi(f_{p_1}) \cdots \varphi(f_{p_n}) \right\rangle - \sum_C \mathcal{C}_{AB}^C(x) \left\langle \mathcal{O}_C(0) \varphi(f_{p_1}) \cdots \varphi(f_{p_n}) \right\rangle \right| \\ & \leq m^{[A]+[B]+n} \sqrt{[A]![B]!} \tilde{K}^{[A]+[B]} \prod_i \sup |\hat{f}_{p_i}| \\ & \times \sup \left(1, \frac{|\vec{p}|_n}{m} \right)^{2([A]+[B])(n+2l+1)+3n} \sum_{\lambda=0}^{2l+n/2} \frac{\log^\lambda \sup(1, \frac{|\vec{p}|_n}{m})}{2^\lambda \lambda!} \\ & \times \frac{1}{\sqrt{\Delta!}} \left(\tilde{K} m |x| \sup \left(1, \frac{|\vec{p}|_n}{m} \right)^{n+2l+1} \right)^\Delta. \end{aligned}$$

Here, there are n spectator fields, $\langle . \rangle$ denote Schwinger functions, and \tilde{K} is a constant depending on n, l . Furthermore, $|\vec{p}|_n$ is defined in eq. (51).

Proof:

Let us begin by defining the “remainder functional” for $D = 0, 1, 2, \dots$ by

$$\begin{aligned} R_D^{\Lambda, \Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0)) &:= \hbar L^{\Lambda, \Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0)) - L^{\Lambda, \Lambda_0}(\mathcal{O}_A(x)) L^{\Lambda, \Lambda_0}(\mathcal{O}_B(0)) - \\ &- \sum_C \mathcal{C}_{AB}^C(x) L^{\Lambda, \Lambda_0}(\mathcal{O}_C(0)), \end{aligned} \quad (113)$$

where the sum is over all C with $[C] \leq D$. The corresponding moments of this functional are written as $\mathcal{R}_{D,n,l}^{\Lambda, \Lambda_0}$. Going through the definitions given in sec. 2, we can write the

remainder in the OPE - with UV-cutoff Λ_0 and IR-cutoff set to $\Lambda = 0$, and still considering the full formal power series in \hbar - as

$$\begin{aligned} & \left\langle \mathcal{O}_A(x) \mathcal{O}_B(0) \hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n) \right\rangle - \sum_C \mathcal{C}_{AB}^C(x) \left\langle \mathcal{O}_C(0) \hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n) \right\rangle = \\ & - \sum_{\substack{I_1 \cup \dots \cup I_j = \{1, \dots, n\}, \\ l, l_1 + \dots + l_j = l}} \hbar^{n+l+1-j} \mathcal{R}_{D, |I_1|, l_1}^{0, \Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0); \vec{p}_{I_1}) \tilde{\mathcal{L}}_{|I_2|, l_2}^{0, \Lambda_0}(\vec{p}_{I_2}) \cdots \tilde{\mathcal{L}}_{|I_j|, l_j}^{0, \Lambda_0}(\vec{p}_{I_j}) \prod_{i=1}^n C^{0, \Lambda_0}(p_i). \end{aligned} \quad (114)$$

Here, $\tilde{\mathcal{L}}_{n,l}^{0, \Lambda_0}(\vec{p})$ are the expansion coefficients of the generating functional $\tilde{L}^{0, \Lambda_0}(\varphi) = -L^{0, \Lambda_0}(\varphi) + \frac{1}{2} \langle \varphi, (C^{0, \Lambda_0})^{-1} \star \varphi \rangle$, without any momentum conservation delta-functions taken out. Thus, we need to estimate the quantities $\mathcal{L}_{n,l}^{0, \Lambda_0}$, the quantities $\mathcal{R}_{D,n,l}^{0, \Lambda_0}$, and the covariances C^{0, Λ_0} . Our bounds on the CAG's without insertions give us

$$|\mathcal{L}_{2n,l}^{0, \Lambda_0}(\vec{p})| \leq m^{4-2n} K^{2n+4l-4} (n+l-1)! \sum_{\lambda=0}^l \frac{\log^\lambda \sup(1, \frac{|\vec{p}|}{m})}{2^\lambda \lambda!} \quad (115)$$

and we also have the trivial bound $C^{0, \Lambda_0}(p) \leq [\sup(|p|, m)]^{-2}$. Thus, what remains is to give bounds on $\mathcal{R}_{D,n,l}^{0, \Lambda_0}$. We have the following lemma about the remainder functional:

Lemma 4.1. Let \mathbb{T}^j be the Taylor operator introduced in eq. (44). Then the remainder functionals satisfy:

$$R_D^{\Lambda, \Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0)) = (1 - \sum_{j=0}^{\Delta} \mathbb{T}^j) \left\{ \hbar L_D^{\Lambda, \Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0)) - L^{\Lambda, \Lambda_0}(\mathcal{O}_A(x)) L^{\Lambda, \Lambda_0}(\mathcal{O}_B(0)) \right\}$$

where $\Delta := D - D'$, $D' = [A] + [B]$. For $\Delta < 0$ the sum is by definition empty.

Proof: Recalling the definition of the “oversubtracted” CAG's with two insertions, we first consider the telescopic sum

$$L^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) = L_D^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) + \sum_{j=0}^D [L_{j-1}^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) - L_j^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B)], \quad (116)$$

where $D < D' = [A] + [B]$. Next, for any $0 \leq j$, we prove the relation

$$L_{j-1}^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) - L_j^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) = \sum_{C: [C]=j} \mathcal{D}^C \{ L_{j-1}^{0, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) \} L^{\Lambda, \Lambda_0}(\mathcal{O}_C). \quad (117)$$

To see this, we make the observation that both sides of the equation obey the same homogeneous FE, and the same boundary conditions, owing to the choice for the boundary

conditions made for the CAG's. Hence they must be equal. In view of our definition of the OPE-coefficients \mathcal{C}_{AB}^C for $[C] < D'$ eq. (45), we conclude that, for $D < D'$, we have

$$\hbar L^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) = \hbar L_D^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) + \sum_{[C] \leq D} \mathcal{C}_{AB}^C L^{\Lambda, \Lambda_0}(\mathcal{O}_C). \quad (118)$$

We now subtract from both sides $L^{\Lambda, \Lambda_0}(\mathcal{O}_A)L^{\Lambda, \Lambda_0}(\mathcal{O}_B)$, and we bring the sum with the OPE coefficients over to the left. Then we get the claim of the lemma for $D < D'$. The case of general D now works by induction. We first observe that, for $\Delta = D - D'$, we have

$$\begin{aligned} & \mathbb{T}^{\Delta+1} \left\{ \hbar L_{D+1}^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) - L^{\Lambda, \Lambda_0}(\mathcal{O}_A)L^{\Lambda, \Lambda_0}(\mathcal{O}_B) \right\} \\ &= - \sum_{[C]=D+1} \mathcal{D}^C \left\{ L^{0, \Lambda_0}(\mathbb{T}^{\Delta+1} \mathcal{O}_A) L^{0, \Lambda_0}(\mathcal{O}_B) \right\} L^{\Lambda, \Lambda_0}(\mathcal{O}_C). \end{aligned} \quad (119)$$

This follows again because both sides satisfy the same linear, homogeneous FE with the same boundary conditions. Next, using the inductive hypothesis, and making trivial re-arrangements in the sums:

$$\begin{aligned} & R_{D+1}^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) = R_D^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) - \sum_{[C]=D+1} \mathcal{C}_{AB}^C L^{\Lambda, \Lambda_0}(\mathcal{O}_C) \\ &= (1 - \sum_{j=0}^{\Delta} \mathbb{T}^j) \left\{ \hbar L_D^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) - L^{\Lambda, \Lambda_0}(\mathcal{O}_A)L^{\Lambda, \Lambda_0}(\mathcal{O}_B) \right\} - \sum_{[C]=D+1} \mathcal{C}_{AB}^C L^{\Lambda, \Lambda_0}(\mathcal{O}_C) \\ &= (1 - \sum_{j=0}^{\Delta+1} \mathbb{T}^j) \left\{ \hbar L_{D+1}^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) - L^{\Lambda, \Lambda_0}(\mathcal{O}_A)L^{\Lambda, \Lambda_0}(\mathcal{O}_B) \right\} \\ &+ \hbar(1 - \sum_{j=0}^{\Delta} \mathbb{T}^j) \left\{ L_D^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) - L_{D+1}^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) \right\} \\ &+ \mathbb{T}^{\Delta+1} \left\{ \hbar L_{D+1}^{\Lambda, \Lambda_0}(\mathcal{O}_A \otimes \mathcal{O}_B) - L^{\Lambda, \Lambda_0}(\mathcal{O}_A)L^{\Lambda, \Lambda_0}(\mathcal{O}_B) \right\} \\ &- \sum_{[C]=D+1} \mathcal{C}_{AB}^C L^{\Lambda, \Lambda_0}(\mathcal{O}_C). \end{aligned} \quad (120)$$

We are now in a position to substitute the formulas (117) and (119) for the second and third term on the right side, together with the definition of \mathcal{C}_{AB}^C for $[C] \geq D'$ for the fourth term. Then the last three terms are seen to cancel out, and we are left with the claim of the lemma. \square

Note that for a function f on \mathbb{R}^4 of differentiability class C^{N+1} , we have the formula

$$(1 - \sum_{j=0}^N \mathbb{T}^j) f(x) = \sum_{|w|=N+1} \frac{x^w}{N!} \int_0^1 (1 - \tau)^N \partial^w f(\tau x) d\tau \quad (121)$$

for the remainder of a Taylor expansion in x carried out to order N . By [8], the functionals $L_D^{\Lambda, \Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0))$ are of differentiability class C^Δ in the variable x , where $\Delta = D - D'$, whereas the functionals $L^{\Lambda, \Lambda_0}(\mathcal{O}_A(x))$ are smooth in x . We write the operator $1 - \sum_{j \leq \Delta} \mathbb{T}^j$

in the statement of the previous lemma as $(1 - \sum_{j \leq \Delta-1} \mathbb{T}^j) - \mathbb{T}^\Delta$, and we rewrite the first operator in parenthesis as a remainder in a Taylor expansion to order $N = \Delta - 1$ as in (121). Then, by the previous lemma and the Lowenstein-rules, we can write the remainder as:

$$\begin{aligned} R_D^{\Lambda, \Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0)) &= \sum_{|w|=\Delta} \left[\frac{x^w}{(\Delta-1)!} \int_0^1 d\tau (1-\tau)^{\Delta-1} \right. \\ &\times \left(\hbar L_D^{\Lambda, \Lambda_0}(\partial^w \mathcal{O}_A(\tau x) \otimes \mathcal{O}_B(0)) - L^{\Lambda, \Lambda_0}(\partial^w \mathcal{O}_A(\tau x)) L^{\Lambda, \Lambda_0}(\mathcal{O}_B(0)) \right) - \\ &\left. \frac{x^w}{w!} \left(\hbar L_D^{\Lambda, \Lambda_0}(\partial^w \mathcal{O}_A(0) \otimes \mathcal{O}_B(0)) - L^{\Lambda, \Lambda_0}(\partial^w \mathcal{O}_A(0)) L^{\Lambda, \Lambda_0}(\mathcal{O}_B(0)) \right) \right] \end{aligned} \quad (122)$$

where $\partial^w \mathcal{O}_A$ on the right side denotes the linear combination of insertions obtained by formally carrying out the differentiations:

$$\partial^w \mathcal{O}_A = \sum_{w_1 + \dots + w_{n'} = w} c_{\{w_i\}} \partial^{w_1 + w'_1} \varphi \dots \partial^{w_{n'} + w'_{n'}} \varphi. \quad (123)$$

Taking the moments of this equation, and setting also $\Lambda = 0$, gives:

$$\begin{aligned} \mathcal{R}_{D, n, l}^{0, \Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0); \vec{p}) &= \sum_{|w|=\Delta} \left[\frac{x^w}{(\Delta-1)!} \int_0^1 d\tau (1-\tau)^{\Delta-1} \right. \\ &\times \left(\mathcal{L}_{n, l-1, D}^{0, \Lambda_0}(\partial^w \mathcal{O}_A(\tau x) \otimes \mathcal{O}_B(0); \vec{p}) - \sum_{\substack{I_1 \cup I_2 = \{1, \dots, n\} \\ l_1 + l_2 = l}} \mathcal{L}_{|I_1|, l_1}^{0, \Lambda_0}(\partial^w \mathcal{O}_A(\tau x); \vec{p}_{I_1}) \mathcal{L}_{|I_2|, l_2}^{0, \Lambda_0}(\mathcal{O}_B(0); \vec{p}_{I_2}) \right) - \\ &\left. \frac{x^w}{w!} \left(\mathcal{L}_{n, l-1, D}^{0, \Lambda_0}(\partial^w \mathcal{O}_A(0) \otimes \mathcal{O}_B(0); \vec{p}) - \sum_{\substack{I_1 \cup I_2 = \{1, \dots, n\} \\ l_1 + l_2 = l}} \mathcal{L}_{|I_1|, l_1}^{0, \Lambda_0}(\partial^w \mathcal{O}_A(0); \vec{p}_{I_1}) \mathcal{L}_{|I_2|, l_2}^{0, \Lambda_0}(\mathcal{O}_B(0); \vec{p}_{I_2}) \right) \right]. \end{aligned}$$

At this stage, we can use our previous bounds on the CAG's to control the remainder.

Using Cor. 3.4, the first term in $[\dots]$ can be bounded by

$$\begin{aligned}
& \left| \sum_{|w|=\Delta} \frac{x^w}{(\Delta-1)!} \int_0^1 d\tau (1-\tau)^{\Delta-1} \mathcal{L}_{n,l-1,D}^{0,\Lambda_0}(\partial^w \mathcal{O}_A(\tau x) \otimes \mathcal{O}_B(0); \vec{p}) \right| \quad (124) \\
& \leq \frac{|x|^\Delta}{(\Delta-1)!} \sum_{|w|=\Delta} \sup_{0 \leq \tau \leq 1} \left| \mathcal{L}_{n,l-1,D}^{0,\Lambda_0}(\partial^w \mathcal{O}_A(\tau x) \otimes \mathcal{O}_B(0); \vec{p}) \right| \\
& \leq \frac{|x|^\Delta}{(\Delta-1)!} m^{D-n} K^{D(n/2+2l-2)^3} \sqrt{(|w'| + \Delta)! |w''|!} \\
& \quad \times \sum_{|w|=\Delta} \sum_{w_1+\dots+w_{n'}=w} c_{\{w_i\}} \sup(1, \frac{|\vec{p}|}{m})^{2D(n+2l-1)} \sum_{\lambda=0}^{2l+n/2-2} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda \lambda!} \\
& \leq m^{D'-n} (K^{(n/2+2l-2)^3} m |x|)^\Delta K^{D'(n/2+2l-2)^3} \\
& \quad \times \sup(1, \frac{|\vec{p}|}{m})^{2D(n+2l-1)} \frac{\sqrt{|w'|! |w''|!}}{\sqrt{\Delta!}} \sum_{\lambda=0}^{2l+n/2-2} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda \lambda!}.
\end{aligned}$$

The last inequality holds for a somewhat larger constant K needed in order to absorb factors Δ , $(4n')^\Delta \leq (4n' + 4n'')^\Delta \leq [4(n + 2l + 1)]^\Delta$ from the sum over w , and $2^{D'+\Delta}$ from $(|w'| + \Delta)! \leq 2^{|w'|+\Delta} |w'|! \Delta! \leq 2^{D'+\Delta} |w'|! \Delta!$. The other three terms in $[\dots]$ can be estimated in the same way using also our estimates for the CAG's with one operator insertion given in Cor. 3.2. They are bounded by an expression of the same form. Putting these straightforward estimates together, and defining also $\tilde{K} := K^{(n/2+2l)^3}$, we thereby demonstrate the following lemma:

Lemma 4.2. Let $D' = [A] + [B]$, $D = D' + \Delta$, $\Delta = 0, 1, 2, \dots$ and $A = \{n', w'\}$, $B = \{n'', w''\}$. The remainder functional satisfies the uniform bound

$$\begin{aligned}
& |\mathcal{R}_{D,n,l}^{0,\Lambda_0}(\mathcal{O}_A(x) \otimes \mathcal{O}_B(0); \vec{p})| \quad (125) \\
& \leq m^{D'-n} (\tilde{K} m |x|)^\Delta \tilde{K}^{D'} \sup(1, \frac{|\vec{p}|}{m})^{2D(n+2l+1)} \frac{\sqrt{|w'|! |w''|!}}{\sqrt{\Delta!}} \sum_{\lambda=0}^{2l+n/2} \frac{\log_+^\lambda(\frac{|\vec{p}|}{m})}{2^\lambda \lambda!}
\end{aligned}$$

with a constant \tilde{K} depending only on n, l .

Substituting the bound stated in the lemma into eq. (114), using the trivial estimate $C^{0,\Lambda_0}(p) \leq [\sup(m, |p|)]^{-2}$, the estimate (115), and the fact that \hat{f}_{p_i} is supported in a ball of radius ϵ around p_i , we get the statement of the theorem for a sufficiently large new constant \tilde{K} . This completes the proof. \square

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